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DEA models equivalent to general Nth order stochastic dominance efficiency tests

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ABSTRACT

We introduce data envelopment analysis (DEA) models equivalent to efficiency tests with respect to the Nth order stochastic dominance (NSD). In particular, we focus on strong and weak variants of convex NSD efficiency and NSD portfolio efficiency. The proposed DEA models are in relation with strong and weak Pareto–Koopmans efficiencies and employ Nth order lower and co-lower partial moments.

1. Introduction and notation

In this paper, we establish a link between data envelopment analysis (DEA) models [7] and Nth order stochastic dominance efficiency tests [13]. Such a link can be very useful because DEA literature provides many results on stability and sensitivity with respect to the input data, algorithmic issues, and methods on ranking efficient units (super-efficiency, cross-efficiency, etc., see [8] for a review) which can be applied to stochastic dominance theory. Moreover, the utility-based interpretation of stochastic dominance relations and efficiency (see [9,11,12,14]) can be used for the proposed DEA models. We generalize results on equivalence obtained for second-order stochastic dominance (SSD) by [5].

The proposed DEA models are derived from NSD efficiency tests introduced in [13]. We show how equivalent DEA models can be obtained using particular directional distance measures, cf. [4], which modify the well-known directional distance function, cf. [6]. The tests by [13] were proposed for the weak variants of the convex NSD efficiency and NSD portfolio efficiency only. Thus, we extend the analysis to the strong efficiencies and we show how these notations differ on a simple example. Note that the optimal solutions of the proposed DEA models are weakly or strongly Pareto–Koopmans efficient.

Let $x_{j,r}$ denote return of asset $j \in \{1, \ldots, M\}$ taken with probability p_r, $r = 1, \ldots, R$. We assume that the columns of $\{x_{j,r}\}$ are sorted in ascending order according to prospect $\tau = (\tau_1, \ldots, \tau_M)$, i.e. if we set $x_r^* = \sum_{j=1}^M x_{j,r}\tau_j$, we have $x_1^* \le x_2^* \le \cdots \le x_R^*$. We denote by $y_1 < \cdots < y_S$ all sorted returns, where $S \le MR$. We use $q_{j,s} = \sum_{r=1}^R p_r \mathbb{I}_{(x_{j,r}=y_s)}$ for all $j \in \{1, \ldots, M\}, s \in \{1, \ldots, S\}$, where $\mathbb{I}_{(\cdot)}$ is equal to one if the condition $\cdot$ is fulfilled and to zero otherwise.

Portfolios are identified by (nonnegative) weights $\lambda = (\lambda_1, \ldots, \lambda_M)$ such that $\sum_{j=1}^M \lambda_j = 1$. The set of all feasible portfolio weights is denoted by Λ. For $n = 0, 1, \ldots,$ we define nth order lower partial moment of the ith asset by

$$\mathrm{LPM}_i^n(w) = \sum_{r=1}^R p_r [w - x_{i,r}]_+^n = \sum_{s=1}^S q_{i,s}[w - y_s]_+^n,$$

and nth order co-lower partial moment [1] by

$$\mathrm{coLPM}_{\tau,\lambda}^n(w) = \sum_{r=1}^R p_r \left(w - \sum_{j=1}^M x_{j,r}\lambda_j\right) \left[w - \sum_{j=1}^M x_{j,r}\tau_j\right]_+^n,$$

where $[y]_+^n = y^n \mathbb{I}_{(y \ge 0)}$. Note that its most important property is linearity with respect to the weights λ_j when portfolio weights τ_j and threshold w are given.

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2. DEA models

We employ DEA models based on directional distance measures. The choice of the (positive) direction is motivated by the paper [15]; see also [4] for a discussion in financial area. Employing a directional distance measure which expresses relative improvement necessary to reach the efficient frontier, see [4], a decision making unit (asset, portfolio) is classified as efficient if and only if the optimal value of the directional distance DEA model is equal to zero.

2.1. Weak Pareto–Koopmans efficiency

Paper [13] considered weak convex NSD efficiency and weak NSD portfolio efficiency.

2.1.1. Weak convex NSD efficiency

Let $U_N = \{u(x) : (-1)^n u^{(n)}(x) \le 0,\ \forall x,\ n = 1, \dots, N\}$ denote a subset of all utility functions, where $u^{(n)}$ is the nth derivative of u. Moreover, set $\bar{x}_j = \sum_{r=1}^{R} p_r x_{j,r},\ \forall j \in \{1, \dots, M\}$.

Definition 2.1 (*[13]*)**.** The ith asset is weakly convex NSD efficient (relative to the set of assets $\{1, \dots, M\}$), $N \ge 1$, if there exists a utility function $u \in U_N$ for which it is preferred to every asset:

$$\sum_{r=1}^{R} p_r u(x_{i,r}) \ge \sum_{r=1}^{R} p_r u(x_{j,r}) \quad \forall j = 1, \dots, M.$$

If such an admissible utility function does not exist the ith asset is called weakly convex NSD inefficient.

Proposition 2.1. *Consider the directional distance DEA model*

$$\begin{aligned}
&\max_{\lambda_j, \theta} \theta \\
&\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \bar{x}_i, \\
&\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^n(y_S) \le \mathrm{LPM}_i^n(y_S), \quad n = 2, \dots, N-2, \\
&\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_k) \le \mathrm{LPM}_i^{N-1}(y_k), \quad k = 1, \dots, S-1, \\
&\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_S) \le \mathrm{LPM}_i^{N-1}(y_S) - \theta \cdot d, \\
&\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,
\end{aligned}$$

with the direction

$$d = \mathrm{LPM}_i^{N-1}(y_S) - \min_j \mathrm{LPM}_j^{N-1}(y_S).$$

If $d > 0$ then ith asset is weakly convex NSD efficient if and only if the optimal value of the directional distance DEA model is equal to zero, that is, the ith asset is DEA efficient. Moreover, if the direction is equal to zero, then the ith asset is weakly convex NSD efficient.

Proof. We employ the problem formulated in [13]; see Theorem 2. Since $y_S \ge x_{j,r},\ \forall j \in \{1, \dots, M\},\ \forall r \in \{1, \dots, R\}$ we obtain for $n = 1$ and for all $j \in \{1, \dots, M\}$:

$$\mathrm{LPM}_j^1(y_S) = \sum_{r=1}^{R} p_r [y_S - x_{j,r}]_+^1 = y_S - \bar{x}_j.$$

The first constraint then easily follows from [13, constraint (23.1)] and the rest of the proof for $d > 0$ is straightforward. If the direction is equal to zero, then the ith asset is weakly convex NSD efficient, because no improvement to the efficient frontier is possible. □

Proposition 2.1 shows that if expected return serves as the output and the lower partial moments given in the constraints as the inputs to the directional distance DEA model then an asset is classified as convex NSD efficient if and only if either it is DEA efficient or the direction is equal to zero. Our model uses a special case of general directional distance function [6], where we consider a directional vector with only one positive element d corresponding to input $\mathrm{LPM}_j^{N-1}(y_S)$.

2.1.2. Weak NSD portfolio efficiency

Weakly convex NSD efficiency generally do not allow for fully diversification across the assets. Therefore we consider also weakly NSD portfolio efficiency. The notation of the employed diversification-consistent DEA models was established by [10] and further investigated by [2,3] in relation with the Pareto–Koopmans efficiency.

Definition 2.2 (*[13]*)**.** The portfolio $\tau \in \Lambda$ is weakly NSD portfolio efficient (relative to Λ), $N \ge 2$, if there exists a utility function $u \in U_N$ for which the portfolio τ is preferred to all portfolios $\lambda \in \Lambda$:

$$\sum_{r=1}^{R} p_r u\left(\sum_{j=1}^{M} x_{j,r} \tau_j\right) \ge \sum_{r=1}^{R} p_r u\left(\sum_{j=1}^{M} x_{j,r} \lambda_j\right) \quad \forall \lambda \in \Lambda.$$

If such an admissible utility function does not exist the portfolio τ is called weakly NSD portfolio inefficient.

Proposition 2.2. *Consider diversification-consistent DEA model based on a directional distance measure*

$$\begin{aligned}
&\max_{\lambda_j, \theta} \theta \\
&\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j, \\
&\mathrm{coLPM}_{\tau,\lambda}^{n-1}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{n-1}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right), \\
&\quad n = 2, \dots, N-2, \\
&\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,k} \tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,k} \tau_j\right), \\
&\quad k = 1, \dots, R-1, \\
&\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right) - \theta \cdot d, \\
&\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,
\end{aligned}$$

with the direction

$$d = \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right) - \min_{\lambda \in \Lambda} \mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,R} \tau_j\right).$$

If $d > 0$ then portfolio τ is weakly NSD portfolio efficient if and only if the optimal value of the diversification-consistent DEA model is equal to zero, that is, portfolio τ is DEA efficient. Moreover, if the direction is equal to zero, then portfolio τ is weakly NSD portfolio efficient.