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# On asymmetric generalization of the Weibull distribution by scale–location mixing of normal laws

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ABSTRACT

Two approaches are suggested to the definition of asymmetric generalized Weibull distribution. These approaches are based on the representation of the two-sided Weibull distributions as variance–mean normal mixtures or more general scale–location mixtures of the normal laws. Since both of these mixtures can be limit laws in limit theorems for random sums of independent random variables, these approaches can provide additional arguments in favor of asymmetric two-sided Weibull-type models of statistical regularities observed in some problems related to stopped random walks, in particular, in problems of modeling the evolution of financial markets



## 1. Introduction. The Weibull distribution

In probability theory and mathematical statistics it is conventional to use the term *Weibull distribution* for a special absolutely continuous probability distribution concentrated on the nonnegative half-line with exponential-power type decrease of the tail. It is called so after the Swedish scientist Waloddi Weibull (1887–1979) who suggested in 1939 to use this distribution in the study of the strength of materials (Weibull, 1939a,b) and thoroughly analyzed this distribution in 1951 (Weibull, 1951) demonstrating good perspectives of the application of this distribution to the description of many observed statistical regularities.

Let $\gamma > 0$. The distribution of the random variable $W_\gamma$:

$$\mathsf{P}\big(W_\gamma < x\big) = \big[1 - e^{-x^\gamma}\big]\mathbf{1}(x \geqslant 0), \quad x \in \mathbb{R}, \tag{1}$$

is called the *Weibull distribution* with shape parameter $\gamma$ (here and in what follows the symbol $\mathbf{1}(C)$ denotes the indicator function of a set $C$). In the subsequent reasoning by $\gamma$ with or without sub- or superscripts we will denote the exponent

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power parameter of the one- or two-sided Weibull distribution that determines the rate of the decrease of the tail(s) of the corresponding Weibull distribution.

However, Weibull was not the first to introduce this distribution. This distribution was for the first time described in 1927 by Fréchet (1927) within the context of the study of the asymptotic behavior of extreme order statistics. Sometimes distribution (1) is called the *Rosin–Rammler distribution* after Paul Rosin and Erich Rammler, German scientists who were the first to use this distribution as a model of statistical regularities in the coal particles sizes in 1933 (Rosin & Rammler, 1933). However, this term also does not completely correspond to the historical truth. In the paper (Stoyan, 2013) Dietrich Stoyan directly writes that this distribution was found by Bennett (1936), Rosin and Rammler (1933) and Rosin, Rammler, and Sperling (1933) within the context of particle size. It is well known that the Weibull distribution family is closed with respect to the operation of taking minimum of independent random variables. As it was demonstrated in Fisher and Tippett (1928) and Fréchet (1927), due to this property the family of Weibull distributions is one of possible limit laws for extreme order statistics. Gnedenko (1943) found necessary and sufficient conditions for the convergence of the distributions of extreme order statistics to the Weibull distribution under linear normalization. Therefore, this distribution is sometimes called the *Weibull–Gnedenko distribution* (Johnson, Kotz, & Balakrishnan, 1994).

The case of small values of the parameter $\gamma \in (0, 1]$ is of special interest for financial and some other applications, since Weibull distributions with such parameters (sometimes called *stretched exponential distributions* Laherrère & Sornette, 1998; Malevergne, Pisarenko, & Sornette, 2005, 2006) occupy an intermediate position between distributions with exponentially decreasing tails (such as exponential and gamma-distributions) and heavy-tailed Zipf–Pareto-type distributions with power-type decrease of tails.

In the paper (Stoyan, 2013) cited above, D. Stoyan notes that the name *exponential power distribution* would better fit to distribution (1), however, the latter term has been occupied by another absolutely continuous distribution with a similar behavior of tails (Box & Tiao, 1973; Grigoryeva & Korolev, 2013; Korolev, Bening, Zaks, & Zeifman, 2012), which, unlike distribution (1), has the exponential power *density*

$$\ell_\gamma(x) = \frac{\gamma}{2\Gamma\left(\frac{1}{\gamma}\right)} e^{-|x|^\gamma}, \quad x \in \mathbb{R},$$

with $\gamma > 0$, whereas distribution (1) has the exponential power *cumulative distribution function*.

In this paper, for definiteness, for distribution (1) we will use the traditional term *Weibull distribution*.

The Weibull distribution is a special generalized gamma-distribution. It is widely used in survival analysis (Elandt-Johnson & Johnson, 1999), in life insurance as a model of the lifetime distribution, in risk insurance as a model of the claim size distribution (Hogg & Klugman, 1984), in economics and financial mathematics as a model of asset returns distribution (DAddario, 1974; Mittnik & Rachev, 1989, 1993) and income distribution (Bartels, 1977; Bordley, McDonald, & Mantrala, 1996), in reliability theory as a model of the distribution of time between failures (Abernethy, 2004; Lawless, 1982), in industrial technology as a model of the distribution of duration of technological stages or time intervals between technological changes (Nawaz Sharif & Nazrul Islam, 1980), in coal industry for the description of statistical regularities of particle sizes (Rosin & Rammler, 1933), in radio engineering and radiolocation, in meteorology, hydrology and many other fields, see, e. g., Abernethy (2004), Johnson and Kotz (1970), Johnson et al. (1994), Kotz and Nadarajah (2000) and Lawless (1982),

In particular, in Mittnik and Rachev (1989) it was discovered that this distribution provides the best fit among others to the observed statistical regularities of the index S&P500, if its positive and negative increments are considered separately thus leading to the concept of a two-sided Weibull distribution with both positive and negative tails decreasing as an exponential power function. Some authors suggested to use the Weibull distribution as the errors distribution in range data modeling (Chen, Gerlach, & Lin, 2008) or the distribution of trading duration (Engle & Russell, 1998).

In Sornette, Simonetti, and Andersen (2000) it was proposed to use a symmetric two-sided Weibull distribution as an unconditional return distribution. A symmetric two-sided Weibull distribution was also mentioned in Malevergne and Sornette (2004), but its properties were not explored. It should be noted that in these papers the attempts to introduce the two-sided Weibull distribution were rather formal and descriptive. In these papers as well as in Chen and Gerlach (2013) the elementary properties of these models were described.

In the present paper, two new approaches are proposed to the definition of general asymmetric two-sided Weibull distribution by representing them as variance–mean and more general scale–location mixtures of normal laws.

Normal mixture representations for the asymmetric two-sided Weibull-type distribution has at least two advantages. First, normal mixtures are limit distributions for sums of a random number of random variables. Therefore, the representability of an asymmetric two-sided Weibull distribution as a normal mixture can give additional grounds for its adequacy in practical problems modeled by stopped random walks, in particular, related to the description of the evolution of financial indexes and thus can provide a deeper insight into these problems. Second, if an asymmetric two-sided Weibull-type distribution is represented as a normal mixture, then to estimate its parameters one can use well-developed techniques oriented to statistical processing normal mixtures such as iterative EM-type procedures or newly proposed efficient grid methods (see, e. g., Korolev & Korchagin, 2014; Korolev & Nazarov, 2010).

The paper is organized as follows. Section 2 contains some auxiliary results dealing with product representations for Weibull-distributed random variables by normally and exponentially distributed random variables. In Section 3 similar