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# Stochastic elasticity of variance with stochastic interest rates

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This paper aims to improve the implied volatility fitting capacity of underlying asset price models by relaxing constant interest rate and constant elasticity of variance and embedding a scaled stochastic setting for option prices. Using multi-scale asymptotics based on averaging principle, we obtain an analytic solution formula of the approximate price for a European vanilla option. The combined structure of stochastic elasticity of variance and stochastic interest rates is compared to the structure of stochastic volatility and stochastic interest rates. The result shows that of the two, the former is more appropriate to fit market data than the latter in terms of convexity of implied volatility surface as time-to-maturity becomes shorter.



## 1. Introduction

This paper studies an extensional work on the pricing of European vanilla options under the stochastic elasticity of variance (SEV) model which was recently proposed by Kim, Lee, Zhu, and Yu (2014). Based on the observation that the variance of elasticity of S&P 500 index fluctuates fast around a mean level, the SEV model has been naturally formulated for the pricing of options and applied to other mathematical finance problems. Refer to Yoon, Kim, and Choi (2013) and Yoon and Kim (2013) for American perpetual options and Yang, Lee, and Kim (2014) for portfolio optimization. The SEV model overcomes both geometrically and dynamically the drawbacks of implied volatility surface structure produced by the Black–Scholes model (Black & Scholes, 1973) and the constant elasticity of variance (CEV) model introduced by Cox (1975), and Cox and Ross (1976). For a review on the drawbacks of implied volatility surface structure of the CEV or local volatility models in general, see Hagan, Kumar, Lesniewski, and Woodward (2002).

Clearly, the SEV model provides a great improvement over the CEV model in view of geometric structure of implied volatilities for options with long time-to-maturities. However, it still does not create sufficiently accurate fit to market for options with short time-to-maturities whose implied volatility presents a special convex shape. From the point of view that the trading volume of short time-to-maturity options is usually much higher than that of long time-to-maturity options, it is important to resolve the convexity issue. This suggests that the SEV model requires to be extended to a model in such a way that it can reflect real market phenomena more effectively. In fact, any stochastic volatility model in which the volatility is driven by a single factor diffusion has this type of trouble fitting implied volatility curves across all time-to-maturities. In general, two factor models are simply not sufficient to describe the dynamics of the underlying price movement. So, it is natural to add a factor in order to increase the fitting capacity. Then the question is how to choose the adding factor to the

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SEV model. A possible choice is adding a jump process to the SEV model or an incorporation of stochastic interest rates into the model. The latter is chosen in this paper because it is relatively easy to obtain an analytic formula of the option price. Also, it usually takes more time to compute the option price under a jump–diffusion model.

Since the first study of Merton (1973), there have been quite a number of unified options and term structure models. Although it has been shown empirically in Amin and Jarrow (1992) and Rindell (1995) that the pricing error of the Black–Scholes model with stochastic interest rates is smaller than that of the model with constant interest rate, the error may not be significant enough to contribute to the convexity of implied volatilities. However, the situation may become different for stochastic volatility models as shown by Kim, Yoon, and Yu (2014) for example. Also, adding stochastic interest rates to a stochastic volatility Lévy model may produce a meaningful contribution to this line of literature, for instance, as given by Sattayatham and Pinkham (2013).

So, based upon a conjecture that stochastic elasticity of variance and stochastic rates of interest are put together to provide a significant contribution to the convexity issue, a combined structure of the SEV model and the stochastic interest rate model of Hull and White (1996), called the SEVHW model in brief, is considered in this paper. Of course, adding a factor is naturally expected to increase the fitting capacity. So, we use the multi-factor model of Kim et al. (2014), in which the Hull–White interest rate process is added to the multi-scale stochastic volatility model of Fouque, Papanicolaou, Sircar, and Solna (2011), as the benchmark model and compare fitting capacity of the benchmark model to ours.

This article is organized as follows. Section 2 formulates a SEV model incorporated with the Hull–White interest rates. Under the SEVHW model, a partial differential equation (PDE) for the price of a European vanilla option is derived by using the Feynman–Kac formula. Section 3 utilizes multi-scale asymptotics based on averaging principle to derive an approximate solution of the option price. Section 4 studies the dynamics of implied volatility surface and calibration to market data and compares the SEVHW model to the benchmark model. The concluding remarks are stated in Section 5.

## 2. Model formulation

We consider an underlying asset price model given by the stochastic differential equation (SDE)

$$dX_t = \mu_t X_t dt + \sigma X_t^{1-\gamma f(Y_t)} dW_t^x,$$
$$dY_t = \alpha(m - Y_t)dt + \beta dW_t^y$$

under a market probability measure, where $\mu_t$ is a stochastic process, $\gamma, m, \alpha$ and $\beta$ are some constants, $f$ is a smooth function satisfying $0 \le c_1 \le f \le c_2 \le \frac{1}{2\gamma}$ for some positive constants $c_1$ and $c_2$ (cf. Karatzas & Shreve, 1991), and $W_t^x$ and $W_t^y$ are correlated standard Brownian motions. The process $Y_t$ is an ergodic process whose typical time to return to the mean level of its long-run distribution is given by the reciprocal of $\alpha$ and its invariant distribution is normal with $N(m, \nu^2)$, where $\nu = \beta/\sqrt{2\alpha}$. Intuitively, if mean reversion rate $\alpha$ goes to infinity, the underlying asset price $X_t$ approaches the CEV diffusion. Also, if $\gamma$ goes to zero, the model approaches the geometric Brownian motion model. So, we introduce two small parameters representing the inverse of mean reversion rate $\alpha$ and the parameter $\gamma$, that is, $\epsilon$ and $\delta$ satisfying $\epsilon = \frac{1}{\alpha}$ and $\delta = \gamma^2$, respectively. Then, under a martingale (risk-neutral) probability measure, the model above is transformed into the SDEs

$$\begin{aligned} dX_t &= r_t X_t dt + \sigma X_t^{1-\eta_t^\delta} dW_t^{x,*}, \qquad \eta_t^\delta = \sqrt{\delta} f(Y_t) \\ dr_t &= (b(t) - a r_t)\, dt + \check{\sigma}\, dW_t^r, \\ dY_t &= \left[\frac{1}{\epsilon}(m - Y_t) - \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\Lambda(X_t, Y_t)\right] dt + \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\, dW_t^{y,*}, \end{aligned} \tag{2.1}$$

where $a$ (mean reversion rate of interest), $\sigma$ (volatility coefficient of underlying asset), $\check{\sigma}$ (volatility of interest rate), $m$ and $\nu$ are constants, and $b(t)$ (average direction of interest rate movement) is a deterministic function of time, and the correlation of Brownian motions $W_t^{x,*}$ and $W_t^{y,*}$ is given by $d\langle W^{x,*}, W^{y,*}\rangle_t = \rho_{xy} dt$, and $W^r$ is correlated with $W^{x,*}$ such that $d\langle W^{x,*}, W^r\rangle_t = \rho_{xr} dt$ but $W_t^r$ is assumed to be independent of $W^{y,*}$. The function $\Lambda$ denotes the risk premium of the elasticity risk. Note that the market price of interest rate risk given by the measure change is absorbed by the term $b(t)$ as in Pelsser (2000).

Now, we define the no-arbitrage price of a European vanilla option by

$$P(t, x, r, y) = E^*\left[e^{-\int_t^T r_s ds} h(X_T) | X_t = x, r_t = r, Y_t = y\right],$$

where $E^*$ denotes expectation with respect to the risk-neutral measure and $h$ is a given payoff function. By using the well-known Feynman–Kac formula (cf. Oksendal, 2003), one can derive a PDE problem for the solution $P^{\epsilon,\delta}(t, x, r, y) := P(t, x, r, y)$ as follows.

$$\mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta}(t, x, r, y) = 0, \quad t < T, \qquad \mathcal{L}^{\epsilon,\delta} := \frac{\partial}{\partial t} + \mathcal{L}^{\epsilon,\delta}_{X,r,Y} - r, \qquad P^{\epsilon,\delta}(T, x, r, y) = h(x), \tag{2.2}$$