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# Noninformative priors for the generalized half-normal distribution

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ABSTRACT

In this paper, we develop noninformative priors for the generalized half-normal distribution when scale and shape parameters are of interest, respectively. Especially, we develop the first and second order matching priors for both parameters. For the shape parameter, we reveal that the second order matching prior is a highest posterior density (HPD) matching prior and a cumulative distribution function (CDF) matching prior. In addition, it matches the alternative coverage probabilities up to the second order. For the scale parameter, we reveal that the second order matching prior is neither a HPD matching prior nor a CDF matching prior. Also, it does not match the alternative coverage probabilities up to the second order. For both parameters, we present that the one-at-a-time reference prior is a second order matching prior. However, Jeffreys' prior is neither a first nor a second order matching prior. Methods are illustrated with both a simulation study and a real data set.



## 1. Introduction

Fatigue is a structural damage which occurs when a material is exposed to stress and tension fluctuations. The half-normal and Birnbaum–Saunders distributions are commonly used to describe the lifetime process under fatigue. Especially they may be preferred for modeling of monotone hazard rates because their shapes are negatively or positively skewed. However, they are not proper for a reasonable parametric fit when one models non-monotone failure rates such as the bathtub shaped and the unimodal failure rates, which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode (Pescim, Clarice, Cordeiro, Ortega, & Urbano, 2010). Recently, Díaz-García and Leiva (2005) proposed a new family of generalized Birnbaum–Saunders distributions based on a contoured elliptical distribution, Cooray and Ananda (2008) proposed a generalized half-normal (GHN) distribution derived from a model for static fatigue, and Pescim et al. (2010) proposed a beta generalized half-normal distribution which contains the half-normal and the GHN distributions. In this paper, we focus on the GHN distributions.

Let $X$ be a random variable with a GHN distribution with the shape parameter $\eta$ and the scale parameter $\beta$. Then the probability density function of $X$ is

$$f(x|\eta,\beta) = \frac{2}{\sqrt{\pi}}\frac{\eta}{x}\left(\frac{x}{\beta}\right)^{\eta}\exp\left\{-\frac{1}{2}\left(\frac{x}{\beta}\right)^{2\eta}\right\}, \quad x > 0, \tag{1}$$

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where $\eta > 0$ and $\beta > 0$. Cooray and Ananda (2008) showed that the GHN distribution exhibits more exclusive mathematical tractability and statistical attractiveness with a more flexible and thicker left tail than the other leading lifetime distributions such as Weibull, gamma, and lognormal distributions. In this paper Bayesian methods are presented for the GHN distribution. Especially, we develop noninformative priors for the GHN distribution.

We consider two noninformative priors: the matching and reference priors. The idea of matching priors is that the coverage probability of a Bayesian credible interval is asymptotically equivalent to that of the frequentist confidence interval up to a certain order (Welch & Peers, 1963). This prior was used in many papers (Datta & Ghosh, 1995a,b; Datta & Ghosh, 1996; Datta & Mukerjee, 2004; DiCiccio & Stern, 1994; Mukerjee & Ghosh, 1997; Mukerjee & Dey, 1993; Mukerjee & Reid, 1999; Stein, 1985; Tibshirani, 1989). Especially, Datta and Mukerjee (2004) provided a thorough and comprehensive discussion of various probability matching criteria. On the other hand, the reference priors maximize the Kullback–Leibler divergence between the prior and the posterior (Bernardo, 1979). Berger and Bernardo (1989, 1992) and Ghosh and Mukerjee (1992) provided a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is successful in various practical problems and the reference priors quite often satisfy the matching criterion described earlier. In this paper, we propose matching priors as well as reference priors for the GHN distribution.

The outline of the remaining sections is as follows. In Section 2, we respectively develop the first order and second order probability matching priors and the reference priors for the shape parameter in GHN. In Section 3, we develop the first order and second order probability matching priors for the scale parameter. We also derive the reference priors for the parameters. Simulation studies and real data analysis are described in Section 4. Finally conclusions and extensions are provided in Section 5.

## 2. The noninformative priors for the shape parameter

### 2.1. The probability matching priors

For a prior $\pi$, let $\theta_1^{1-\alpha}(\pi;\mathbf{X})$ denote the $(1-\alpha)$th quantile of the posterior distribution of $\theta_1$, that is,

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{X})|\mathbf{X}] = 1-\alpha, \tag{2}$$

where $\theta_1$ is the parameter of interest and $P^{\pi}$ is a probability measure on posterior distribution under the prior $\pi$. Let $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_t)$. We want to find priors $\pi$ for which

$$P_{\boldsymbol{\theta}}[\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{X})] = 1-\alpha + o(n^{-r}), \tag{3}$$

for some $r > 0$, as $n$ goes to infinity. Here $P_{\boldsymbol{\theta}}$ is a frequentist's probability measure. Priors $\pi$ satisfying (3) are called *matching priors*. If $r = 1/2$, then $\pi$ is referred to as a first order matching prior, while if $r = 1$, $\pi$ is referred to as a second order matching prior.

In order to find such matching priors $\pi$, let

$$\theta_1 = \eta \quad \text{and} \quad \theta_2 = \beta \exp\left\{\frac{c_1}{2\eta}\right\},$$

where $c_1 = \log 2 + \gamma_1$ and $\gamma_1 = \int_0^{\infty} (\log y) \frac{2}{\sqrt{\pi}} y^{\frac{1}{2}} \exp\{-y\} dy$. The likelihood function for the parameters $\boldsymbol{\theta} = (\theta_1,\theta_2)$ in the model (1) is given by

$$L(\boldsymbol{\theta}) \propto \frac{\theta_1}{x}\left(\frac{x}{\theta_2}\right)^{\theta_1} \exp\left\{-\frac{1}{2}\left(\frac{x}{\theta_2}\right)^{2\theta_1} e^{c_1}\right\}. \tag{4}$$

Based on (4), the Fisher information matrix is given by

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{2+\gamma_2-\gamma_1^2}{2}\dfrac{1}{\theta_1^2} & 0 \\ 0 & \dfrac{2\theta_1^2}{\theta_2^2} \end{pmatrix}, \tag{5}$$

where $\gamma_i = \int_0^{\infty} (\log y)^i \frac{2}{\sqrt{\pi}} y^{\frac{14}{2}} \exp\{-y\} dy$, $i = 1, 2$. From the above Fisher information matrix $\mathbf{I}$, $\theta_1$ is orthogonal to $\theta_2$ in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of first order probability matching priors is characterized by

$$\pi_m^{(1)}(\theta_1,\theta_2) \propto \theta_1^{-1} d(\theta_2), \tag{6}$$

where $d(\theta_2) > 0$ is an arbitrary function differentiable in its argument. The class of priors given in (6) can be narrowed down to the second order probability matching priors as given in Mukerjee and Ghosh (1997). Let $L \equiv L(\boldsymbol{\theta})$, and define