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Inference in affine shape theory under elliptical models

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ABSTRACT

This paper studies the elliptical statistical affine shape theory under certain particular conditions on the evenness or oddness of the number of landmarks. In such a case, the related distributions are polynomials, and the inference is easily performed; as an example, a landmark data is studied, and the performance of the polynomial density versus the usual series density is compared.

1. Introduction

Various disciplines such as image analysis, biology and medicine, are interested in measuring, describing and comparing the shapes of objects. It can be assumed that the objects are summarized by a finite number of points, called landmarks, which can be used as references for carrying out matches between and within populations of objects. The mathematical and statistical matching problem has been studied by several authors; see Kendall, Barden, Carne, and Le (1999) for an excellent topological and geometrical characterization of the shape space. As a consequence of this geometrical understanding of shape, the procrustes theory, including projective shape analysis and its relation to affine shape analysis, has been studied profusely in applications; some works in these directions are due to Domokos and Kato (2010), Ecaberth and Thiran (2004), Glasbey and Mardia (2001), Goodall (1991), Groisser and Tagare (2009), Horgan, Creasey, and Fenton (1992), Kent, Mardia, and Taylor (2004), Lin and Fang (2007), Lindeberg and Garding (1997), Mai, Chang, and Hung (2011), Mardia, Goodall, and Walder (1996), Mardia and Patrangenaru (2005), Mardia, Patrangenaru, and Sugathadasa (2005), Mokhtarian and Abbasi (2002) and Patrangenaru and Mardia (2003), among many others.

Now, from the statistical point of view, shape analysis is concerned with methods for analyzing shapes in the presence of randomness, when the objects can be sampled at random from the population. The statistical approach estimates the population mean shape and the structure of the population shape variability and performs inferences on these population quantities; some important studies and compilations of works in this field can be found in Dryden and Mardia (1998) and Small (1996), and the references therein.

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There are two main approaches to affine shape theory: the statistical shape analysis and the affine shape analysis in computer vision. The affine shape distribution based on a Gaussian model has been studied by Goodall and Mardia (1993) (corrected by Caro-Lopera, Díaz-García, & González-Farías, 2009a and Díaz-García, Gutiérrez, & Ramos-Quiroga, 2003), Berthilsson and Heyden (1999) and Leung, Burl, and Perona (1998) (the densities obtained there can be seen as particular cases of the general densities derived by Caro-Lopera et al., 2009a). On the other hand, the affine shapes in computer vision have been studied by various, such as Heyden (1996) and Sparr (1996). Fortunately, Patrangenaru and Mardia (2003) proposes a unified approach, proving that affine shapes in statistical shape analysis are equivalent to affine shapes in computer vision.

The core of the statistical or mathematical problems resides in the fact that some information about the *figure* must be filtered out, according to the phenomena under consideration, by removing some kind of noise through translation, scale, rotation, reflection and/or uniform shear, in order to obtain the *shape* of the object. Strictly speaking, Goodall and Mardia (1993) defined the affine shape distributions as the marginal distributions modulo affine transformations; then the addressed unified definition of affine shape is based on the following definition.

Definition 1. Two figures $\mathbf{X} : N \times K$ and $\mathbf{X}_1 : N \times K$ have the *same configuration*, or *affine shape*, if $\mathbf{X}_1 = \mathbf{XE} + \mathbf{1}_N\mathbf{e}'$, for some translation $\mathbf{e} : K \times 1$ and a nonsingular $\mathbf{E} : K \times K$.

From Ecaberth and Thiran (2004) the nonsingular matrix $\mathbf{E}$ can be seen as the product of elementary matrices, say $\mathbf{E} = \mathbf{E}_1 \cdots \mathbf{E}_m$, corresponding to some m elementary geometric operations, including rotations, non-uniform scalings and/or shearings. In our case, $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$, involving elementary geometric operations of rotations (via the orthogonal matrix $\mathbf{H}$), and uniform scaling shearing (via the positive definite matrix $\mathbf{F}^{1/2} > \mathbf{0}$). For further readings in the main elements of transformations and the geometry of shape, and some graphical examples, the reader is referred to Dryden and Mardia (1998) and Ecaberth and Thiran (2004).

Now, let $\mathbf{L}$ be an $N - 1 \times N$ Helmert[1] sub-matrix and consider $\mathbf{Y} = (\mathbf{Y}_1' \mid \mathbf{Y}_2')'$, where $\mathbf{Y}_1$ is a $K \times K$ nonsingular matrix and $\mathbf{Y}_2$ a $q \times K$ matrix, with $q = N - K - 1 \geq 1$ and $N > K + 1$; otherwise if $N \leq K + 1$, only a configuration is obtained.

Then, the $N - 1 \times K$ configuration matrix $\mathbf{U}$, which contains the configuration coordinates of the original $N \times K$ landmark matrix $\mathbf{X}$, is constructed in two steps summarized in the expression

$$\mathbf{LX} = \mathbf{Y} = \mathbf{UE}. \tag{1}$$

Note that $\mathbf{E} = \mathbf{Y}_1$ is nonsingular with probability 1 and $\mathbf{U}$ have the form $\mathbf{U} = (\mathbf{I} \mid \mathbf{V}')'$, where $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_1^{-1}$. Following Goodall and Mardia (1993), configuration coordinates can also be defined when $\mathbf{Y}_1$ is singular; there are two cases: if $\mathbf{L}$ can be set, the identity matrix in $\mathbf{U}$ is replaced by a diagonal matrix with the first m diagonal entries 1 and the remaining elements 0, where $m = \operatorname{rank}(\mathbf{Y}_1) \leq K$; if $\mathbf{L}$ is fixed, as usual, $\mathbf{U} = (u_{i,j})$ is such that $u_{j^*,k} = 0$ with $k \neq j$ and $u_{j^*,j} = 1$, where $1 \leq j \leq K$, $j^* = \arg\min_i u_{i,j} \neq 0$. See Dryden and Mardia (1998) and Ecaberth and Thiran (2004), for some graphical examples.

In the literature it is generally assumed that $\mathbf{X}$ has an isotropic matrix Gaussian distribution with a mean $\boldsymbol{\mu}_\mathbf{X}$,

$$\mathbf{X} \sim \mathcal{N}_{N\times K}(\boldsymbol{\mu}_\mathbf{X}, \sigma^2\mathbf{I}_N \otimes \mathbf{I}_K),$$

then, under (1), the procedure pursues the distribution of configuration matrix $\mathbf{U}$, which is termed the *configuration or affine distribution*; see Goodall and Mardia (1993).

Later, Caro-Lopera et al. (2009a); Caro-Lopera, Díaz-García, and González-Farías (2009b) generalized this approach assuming that $\mathbf{X}$ have a general elliptically contoured distribution, namely

$$\mathbf{X} \sim \mathcal{E}_{N\times K}(\boldsymbol{\mu}_\mathbf{X}, \boldsymbol{\Sigma}_\mathbf{X} \otimes \boldsymbol{\Theta}_\mathbf{X}, h),$$

and then, they obtained an expression for the configuration or affine distribution in terms of an infinite series of zonal polynomials.

In this way the inference procedure is solved, but only from a theoretical point of view because, even when the zonal polynomials can be computed very quickly, the problem now resides in the convergence and the truncation of the series of zonal polynomials (Koev & Edelman, 2006, p. 845), Caro-Lopera et al. (2009a,b).

Under certain conditions, this article derives the configuration density, based on an elliptical model, as a polynomial of low degree. Section 2 describes a revision of the main definitions, including the affine technique and the necessary mathematical tools to obtain the main result of the paper. Then, Section 3 derives the polynomial configuration distribution under an elliptical model and some necessary conditions on the number of landmarks. In Section 4 the inference procedure in affine shape theory is proposed, and finally in Section 5, the main result is applied to a published data set; a numerical performance comparison between the polynomial density versus the usual infinite series densities of zonal polynomials is made.

[1] The Helmert matrix is a $k \times k$ orthogonal matrix, with its first row of elements equal to $1/\sqrt{k}$, and the remaining rows (which constitute the Helmert sub-matrix $\mathbf{L}$) orthogonal to the first row. The i-th row of the $(k-1) \times k$ matrix $\mathbf{L}$ consists of $\mathbf{L}_i = -1/\sqrt{i(i+1)}$ repeated i times, followed by $-i\mathbf{L}_i$ and then $k - i - 1$ zeros, $i = 1, \ldots, k - 1$.