Journal of Multivariate Analysis 152 (2016) 15–27

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Kummer and gamma laws through independences on trees—Another parallel with the Matsumoto–Yor property

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ARTICLE INFO

Article history:
Received 6 November 2015
Available online 3 August 2016

AMS subject classifications:
60E05
62H05

ABSTRACT

The paper develops a rather unexpected parallel to the multivariate Matsumoto–Yor (MY) property on trees considered in Massam and Wesołowski (2004). The parallel concerns a multivariate version of the Kummer distribution, which is generated by a tree. Given a tree of size p, we direct it by choosing a vertex, say r, as a root. With such a directed tree we associate a map Φ_r. For a random vector $\mathbf{S}$ having a p-variate tree-Kummer distribution and any root r, we prove that $\Phi_r(\mathbf{S})$ has independent components. Moreover, we show that if $\mathbf{S}$ is a random vector in $(0,\infty)^p$ and for any leaf r of the tree the components of $\Phi_r(\mathbf{S})$ are independent, then one of these components has a Gamma distribution and the remaining $p-1$ components have Kummer distributions. Our point of departure is a relatively simple independence property due to Hamza and Vallois (2016). It states that if X and Y are independent random variables having Kummer and Gamma distributions (with suitably related parameters) and $T:(0,\infty)^2\to(0,\infty)^2$ is the involution defined by $T(x,y)=(y/(1+x), x+xy/(1+x))$, then the random vector $T(X,Y)$ has also independent components with Kummer and gamma distributions. By a method inspired by a proof of a similar result for the MY property, we show that this independence property characterizes the gamma and Kummer laws.

1. Introduction

Let X and Y be independent random variables. There are several well-known cases where $U=\phi(X,Y)$ and $V=\psi(X,Y)$ are also independent. A number of distributions have actually been characterized this way. Classical results along these lines include Bernstein's characterization [4] of the Gaussian distribution through independence of $U=X-Y$ and $V=X+Y$, and Lukacs' characterization [18] of the Gamma distribution through independence of $U=X/Y$ and $V=X+Y$.

At the end of the 1990s, a new result of this kind, called the Matsumoto–Yor (MY) property was discovered; see, e.g., [24, p. 43]. It states that if X has a generalized inverse Gaussian (GIG) distribution and Y is Gamma, the random variables $U=1/(X+Y)$ and $V=1/X-1/(X+Y)$ are independent. It arose in studies [22,23] of the conditional structure of some functionals of the geometric Brownian motion. See [17] for a related characterization of the GIG and Gamma distributions through the independence of X and Y and of U and V.

The MY property is also strongly rooted in classical multivariate analysis. Its matrix-variate version appears naturally in the conditional structure of Wishart matrices; see, e.g., [20] as well as [6,9]. A higher-dimensional version of the MY property and related characterization was studied in [19], where a Gamma-type multivariate distribution was obtained by connecting its density shape to a tree. Through a suitable transformation related to directed trees, a random vector having

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http://dx.doi.org/10.1016/j.jmva.2016.07.004

the latter distribution was mapped to independent components with GIG and Gamma distributions. This approach also led to a characterization of the product of GIG and Gamma distributions. In the special case of a chain with two vertices these results are equivalent to the characterization through the original MY property.

The MY property attracted a lot of attention in the last 15 years. In particular, [15] tried to identify all possible functions f and distributions of independent X and Y such that $f(X+Y)$ and $f(X)-f(X+Y)$ are also independent. The MY property corresponds to the case $f(x)=1/x$. Another important case identified in that paper occurs when $f(x)=\ln(1+1/x)$. Thus if X and Y are independent with Kummer and Gamma distributions, then

$$U=X+Y \quad \text{and} \quad V=\frac{1+1/(X+Y)}{1+1/X}$$

are independent Kummer and Beta distributed random variables, respectively.

Characterizations by independence of X and Y and of U and V were obtained in [14,15]. To derive their results, however, the authors needed to impose technical conditions of differentiability or local integrability of logarithms of strictly positive densities. Recently a regression characterization under natural integrability condition was given in [27] without any assumptions on the densities.

In the present paper we are interested in an independence property discovered recently in [10]. It states that if X and Y are independent random variables with Kummer and Gamma distributions, then

$$U=Y/(1+X) \quad \text{and} \quad V=X\,\frac{1+X+Y}{1+X}$$

are also independent and have Kummer and Gamma distributions, respectively. In studying this property, we will exploit several of the ideas described above. In particular, we will give a characterization of the Gamma and Kummer distributions through the independence of the components in the pairs (X,Y) and (U,V). In the proof, inspired from [26], we will use the method of functional equations for densities assuming local integrability of their logarithms. This is reported in Section 2.

Next, we will introduce and study multivariate versions of the property described above. Our approach parallels the one adopted in [19] for a multivariate version of the MY property. We will first define a p-variate tree-Kummer distribution from an undirected tree T of size p. For each vertex r of T, we will define the directed tree by choosing r as its root. To each such directed tree, we will associate a transformation $\Phi_r : (0,\infty)^p \to (0,\infty)^p$ and show that if a random vector $\mathbf{S}$ has a tree-Kummer distribution, then $\Phi_r(\mathbf{S})$ has independent components with Gamma and Kummer distributions. This analogue of Theorem 3.1 in [19] is given in Section 3.

In Section 4 we will derive a characterization of products of Gamma and Kummer distributions (and thus of the tree-Kummer distribution) assuming that for any leaf r of the tree T, the components of $\Phi_r(\mathbf{S})$ are independent. This result parallels Theorem 4.1 in [19]. Finally, Section 5 contains some concluding remarks.

2. Kummer and gamma characterization

The Kummer distribution $\mathcal{K}(\alpha,\beta,\gamma)$ with parameters $\alpha,\gamma>0$, $\beta\in\mathbb{R}$ has density

$$f(x)\propto \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,e^{-\gamma x}\,\mathbb{1}_{(0,\infty)}(x).$$

If $\beta>0$ it is a natural exponential family generated by the second kind Beta distribution. More information on the Kummer distribution, its properties and applications can be found in [2,1,7,8,12,15,16] and in the monograph [3].

By the Gamma distribution $\mathcal{G}(\alpha,\gamma)$ with parameters $\alpha,\gamma>0$, we mean the distribution whose density is given by

$$g(x)\propto x^{\alpha-1}e^{-\gamma x}\mathbb{1}_{(0,\infty)}(x).$$

Consider two independent random variables X and Y with respective distributions $X\sim\mathcal{K}(a,b-a,c)$ and $Y\sim\mathcal{G}(b,c)$, where $a,b,c>0$. Define a bijection $T:\ (0,\infty)^2\to(0,\ \infty)^2$ by

$$T(x,y)=\left(\frac{y}{1+x},\,x\left(1+\frac{y}{1+x}\right)\right).$$

Let $(U,V)=T(X,Y)$. It has been observed in [10] that U and V are independent and $U\sim\mathcal{K}(b,a-b,c)$, $V\sim\mathcal{G}(a,c)$. Our objective in this section is to give a converse of this result, that is a characterization of the Kummer and the Gamma distribution through the independence property mentioned above. Unfortunately, as in the case of the characterization of the Kummer and Gamma distributions obtained by [14,15], we also need to impose some regularity conditions on densities.

Theorem 1. *Let X and Y be two independent positive random variables with positive and continuously differentiable densities on $(0,\infty)$. Suppose that*

$$U=\frac{Y}{1+X},\qquad V=X\left(1+\frac{Y}{1+X}\right),$$

are independent. Then there exist constants $a,b,c>0$, such that $X\sim\mathcal{K}(a,b-a,c)$, $Y\sim\mathcal{G}(b,c)$ or, equivalently, $U\sim\mathcal{K}(b,a-b,c)$ and $V\sim\mathcal{G}(a,c)$.