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On the uniform consistency of the zoonid depth[☆]

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ABSTRACT

Under some mild conditions on probability distribution P, if $\lim_n P_n = P$ weakly then the sequence of zonoid depth functions with respect to P_n converges uniformly to the zonoid depth function with respect to P.

1. Introduction

The *depth* of a point in $\mathbb{R}^d$ with respect to a probability distribution in the same Euclidean space quantifies the degree of centrality of the point with respect to the distribution. A (possibly non-unique) point of maximal depth is, in some sense, central with regard to the given distribution, while depth decreases along rays emanating from that center. The lack of a natural order in the multivariate Euclidean space together with the possibility of introducing a center-outward ordering of data points based on their depths have provided data depth notions with quite some attention from the multivariate statistics community.

Among the numerous notions of data depth introduced in the statistical literature in the last decades, see [1,8,11,15] for particular notions of data depth and some of their applications, the *zonoid depth* [3,4] occupies a prominent position, right after the best known data depths, which are Tukey's halfspace depth [13,14] and Liu's simplicial depth [6].

The *zonoid depth* with regard to the empirical probability of a sample of size n assigns depth 1 to the average of all points from the data cloud, and depth at least k/n to all points that can be obtained as either averages of k points from the data cloud or as a convex combination of a set of such averages. For a general population distribution, the zonoid depth is commonly introduced in terms of its level sets, which are known as *zonoid trimmed regions*. These zonoid trimmed regions can be as well obtained as a transformation of the so-called *lift zonoid* [5] of a distribution P, which is a convex body in $\mathbb{R}^{d+1}$ that characterizes probability distributions with finite first moment.

The uniform consistency of the empirical depth proves to be relevant when establishing a multivariate order with regard to a data cloud. The reason is that the interest is not only to estimate the depth of a point, but the whole depth function,

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see Remark A.3 [15] for a brief discussion. Among other particular applications, it has been used in Theorem 6.1 [9] to prove the consistency of a sample quality index built to compare two distributions, and in Proposition 3.1 [7] to support some distribution-free control charts based on ranks induced by a notion of data depth.

The strong uniform consistency of the empirical halfspace depth was established in pp. 1816–1817 [2], while the one of the simplicial depth under the assumption of absolute continuity of the population distribution in Theorem 5 [6]. As for other depth functions, the uniform consistency of the empirical Mahalanobis depth holds for distributions with bounded second absolute moment, and the one of the Majority depth for elliptical distributions, see Remark 2.2 [9]. The pointwise strong consistency of the zonoid depth was established in Theorem 7.1 (iii) [4]. In the current note, we prove the strong uniform consistency of the zonoid depth.

The paper is organized as follows: Section 2 is devoted to some preliminaries about convex geometry and zonoid depth, while our main result is presented in Section 3.

2. Preliminaries

A *convex body* is a compact convex set in the d-dimensional Euclidean space with nonempty interior. The unit sphere in $\mathbb{R}^d$ is represented by S^{d-1}, while $\langle\cdot,\cdot\rangle$ stands for the scalar product.

Given a convex body $K\subset\mathbb{R}^d$, its *support function*, $h_K : S^{d-1}\mapsto\mathbb{R}$ and, subject to $0\in K$, its *radius-vector function* $\rho_K : S^{d-1}\mapsto\mathbb{R}$ (see [10]) are respectively given by

$$h_K(u)=\sup\{\langle x,u\rangle : x\in K\},$$
$$\rho_K(u)=\sup\{t\ge 0 : tu\in K\}.$$

The *Hausdorff distance* between two convex bodies $K_1,K_2\subset\mathbb{R}^d$ is

$$d_H(K_1,K_2)=\sup_{u\in S^{d-1}}|h_{K_1}(u)-h_{K_2}(u)|.$$

The standard notion of convergence for convex bodies is the Hausdorff one. We say that the sequence of d-dimensional convex bodies $\{K_n\}_n$ converges to the convex body K in the Hausdorff distance if $\lim_n d_H(K_n,K)=0$.

All probability measures P considered hereafter are defined on the general d-dimensional Euclidean space equipped with the Borel σ-algebra and are assumed to have finite first moment, that is, $\int_{\mathbb{R}^d}\|x\|\mathrm{d}P(x)<\infty$.

The *lift zonoid* of P is a convex body in $\mathbb{R}^{d+1}$ containing the origin of coordinates and given by

$$\mathrm{Z}(P)=\left\{\left(\int g(y)\mathrm{d}P(y),\int yg(y)\mathrm{d}P(y)\right),\ \text{such that } g:\mathbb{R}^d\mapsto[0,1]\text{ measurable}\right\}.$$

The *zonoid depth* of $x\in\mathbb{R}^d$ with respect to P is

$$\mathrm{ZD}(x;P)=\sup\{\alpha\in(0,1] : x\in\alpha^{-1}\mathrm{proj}_\alpha(\mathrm{Z}(P))\},$$

where $\mathrm{proj}_\alpha(\mathrm{Z}(P))$ is the projection of the intersection of $\mathrm{Z}(P)$ with the hyperplane $\{(\alpha,x) : x\in\mathbb{R}^d\}$ to the last d coordinates. After multiplication by α^{-1} this set is commonly referred to as *zonoid trimmed region of level* α *of* P and denoted by $\mathrm{ZD}^\alpha(P)=\alpha^{-1}\mathrm{proj}_\alpha(\mathrm{Z}(P))$. The family of convex bodies $\{\mathrm{ZD}^\alpha(P)\}_{\alpha\in(0,1]}$ is nested and decreasing on α, while $\mathrm{ZD}^0(P)$ is defined as the closed and convex set $\mathrm{cl}(\cup_{\alpha>0}\mathrm{ZD}^\alpha(P))$.

If a sequence of probability measures $\{P_n\}_n$ converges weakly to P, then

- $\lim_n\mathrm{Z}(P_n)=\mathrm{Z}(P)$ in the Hausdorff sense (Theorem 3.3 [5]);
- $\lim_n\mathrm{ZD}^\alpha(P_n)=\mathrm{ZD}^\alpha(P)$ in the Hausdorff sense for any $\alpha\in(0,1]$ (Theorem 5.2 (i) [4]);
- if x lies in the interior of the convex hull of the support of P, then $\lim_n\mathrm{ZD}(x;P_n)=\mathrm{ZD}(x;P)$ (Theorem 7.1(iii) [4]).

3. Main result

The zonoid depth of $x\in\mathbb{R}^d$ with respect to P is the supremum of all $0<\alpha\le 1$ such that $(\alpha,\alpha x)$ lies in the lift zonoid of P, and it is thus possible to relate the zonoid depth with the radius-vector function of the lift zonoid. For simplicity, for $x\in\mathbb{R}^d$, we will hereafter write $\mathbf{x}=(1,x)\in\mathbb{R}^{d+1}$ and $u(\mathbf{x})=\mathbf{x}/\|\mathbf{x}\|\in S^d$.

Lemma 3.1. *For any $x\in\mathbb{R}^d$, we have*

$$\mathrm{ZD}(x;P)=\rho_{\mathrm{Z}(P)}(u(\mathbf{x}))\|\mathbf{x}\|^{-1}\le\rho_{\mathrm{Z}(P)}(u(\mathbf{x})).\tag{1}$$

Proof. For any $x\in\mathbb{R}^d$, we have $\alpha x\in\mathrm{proj}_\alpha(\mathrm{Z}(P))$ as long as $\alpha\mathbf{x}=(\alpha,\alpha x)\in\mathrm{Z}(P)$. Finally and since the radius-vector function is only defined on S^{d-1}, we normalize $\mathbf{x}$ in order to obtain

$$\mathrm{ZD}(x;P)=\sup\{\alpha\in(0,1] : (\alpha,\alpha x)\in\mathrm{Z}(P)\}=\rho_{\mathrm{Z}(P)}(u(\mathbf{x}))\|\mathbf{x}\|^{-1}.$$

The inequality in (1) follows from $\|\mathbf{x}\|\ge 1$ for all $x\in\mathbb{R}^d$. □