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Asymptotic inference for a stochastic differential equation with uniformly distributed time delay

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ABSTRACT

For the affine stochastic delay differential equation

$$\mathrm{d}X(t) = a\int_{-1}^{0} X(t+u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t), \quad t \geqslant 0,$$

the local asymptotic properties of the likelihood function are studied. Local asymptotic normality is proved in case of $a \in \left(-\frac{\pi^2}{2}, 0\right)$, local asymptotic mixed normality is shown if $a \in (0, \infty)$, periodic local asymptotic mixed normality is valid if $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$, and only local asymptotic quadraticity holds at the points $-\frac{\pi^2}{2}$ and 0. Applications to the asymptotic behaviour of the maximum likelihood estimator $\widehat{a}_T$ of a based on $(X(t))_{t\in[0,T]}$ are given as $T \to \infty$.

1. Introduction

Assume $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process, $a \in \mathbb{R}$ is a real number, and $(X^{(a)}(t))_{t\in\mathbb{R}_+}$ is a solution of the affine stochastic delay differential equation (SDDE)

$$\begin{cases} \mathrm{d}X(t) = a\displaystyle\int_{-1}^{0} X(t+u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-1, 0], \end{cases} \tag{1}$$

where $(X_0(t))_{t\in[-1,0]}$ is a continuous stochastic process independent of $(W(t))_{t\in\mathbb{R}_+}$. The SDDE (1) can also be written in the integral form

$$\begin{cases} X(t) = X_0(0) + a\displaystyle\int_0^t \int_{-1}^{0} X(s+u)\,\mathrm{d}u\,\mathrm{d}s + W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-1, 0]. \end{cases} \tag{2}$$

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Eq. (1) is a special case of the affine stochastic delay differential equation

$$\begin{cases} \mathrm{d}X(t) = \int_{-r}^{0} X(t+u)\, m_\theta(\mathrm{d}u)\, \mathrm{d}t + \mathrm{d}W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0], \end{cases} \tag{3}$$

where $r > 0$, and for each $\theta \in \Theta$, m_θ is a finite signed measure on $[-r, 0]$, see Gushchin and Küchler (2003). In that paper local asymptotic normality has been proved for stationary solutions. In Gushchin and Küchler (1999), the special case of (3) has been studied with $r = 1$, $\Theta = \mathbb{R}^2$, and $m_\theta = a\delta_0 + b\delta_{-1}$ for $\theta = (a, b)$, where δ_x denotes the Dirac measure concentrated at $x \in \mathbb{R}$, and they described the local properties of the likelihood function for the whole parameter space $\mathbb{R}^2$.

The solution $(X^{(a)}(t))_{t\in\mathbb{R}_+}$ of (1) exists, is pathwise uniquely determined and can be represented as

$$X^{(a)}(t) = x_{0,a}(t)X_0(0) + a\int_{-1}^{0}\int_{u}^{0} x_{0,a}(t+u-s)X_0(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t x_{0,a}(t-s)\,\mathrm{d}W(s), \tag{4}$$

for $t \in \mathbb{R}_+$, where $(x_{0,a}(t))_{t\in[-1,\infty)}$ denotes the so-called fundamental solution of the deterministic homogeneous delay differential equation

$$\begin{cases} x(t) = x_0(0) + a\int_0^t \int_{-1}^{0} x(s+u)\,\mathrm{d}u\,\mathrm{d}s, & t \in \mathbb{R}_+, \\ x(t) = x_0(t), & t \in [-1, 0], \end{cases} \tag{5}$$

with initial function

$$x_0(t) := \begin{cases} 0, & t \in [-1, 0), \\ 1, & t = 0. \end{cases}$$

In the trivial case of $a = 0$, we have $x_{0,0}(t) = 1$, $t \in \mathbb{R}_+$, and $X^{(0)}(t) = X_0(0) + W(t)$, $t \in \mathbb{R}_+$. In case of $a \in \mathbb{R} \setminus \{0\}$, the behaviour of $(x_{0,a}(t))_{t\in[-1,\infty)}$ is connected with the so-called characteristic function $h_a : \mathbb{C} \to \mathbb{C}$, given by

$$h_a(\lambda) := \lambda - a\int_{-1}^{0} \mathrm{e}^{\lambda u}\,\mathrm{d}u, \quad \lambda \in \mathbb{C}, \tag{6}$$

and the set Λ_a of the (complex) solutions of the so-called characteristic equation for (5),

$$\lambda - a\int_{-1}^{0} \mathrm{e}^{\lambda u}\,\mathrm{d}u = 0. \tag{7}$$

Applying usual methods (e.g., argument principle in complex analysis and the existence of local inverses of holomorphic functions), one can derive the following properties of the set Λ_a, see, e.g., Reiss (2002). We have $\Lambda(a) \neq \emptyset$, and $\Lambda(a)$ consists of isolated points. Moreover, $\Lambda(a)$ is countably infinite, and for each $c \in \mathbb{R}$, the set $\{\lambda \in \Lambda_a : \mathrm{Re}(\lambda) \geqslant c\}$ is finite. In particular,

$$v_0(a) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \Lambda_a\} < \infty.$$

Put

$$v_1(a) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \Lambda_a,\ \mathrm{Re}(\lambda) < v_0(a)\},$$

where $\sup\emptyset := -\infty$. We have the following cases:

(i) If $a \in \left(-\frac{\pi^2}{2}, 0\right)$ then $v_0(a) < 0$;
(ii) If $a = -\frac{\pi^2}{2}$ then $v_0(a) = 0$ and $v_0(a) \notin \Lambda_a$;
(iii) If $a \in \left(-\infty, -\frac{\pi^2}{2}\right)$ then $v_0(a) > 0$ and $v_0(a) \notin \Lambda_a$;
(iv) If $a \in (0, \infty)$ then $v_0(a) > 0$, $v_0(a) \in \Lambda_a$, $m(v_0(a)) = 1$ (where $m(v_0(a))$ denotes the multiplicity of $v_0(a)$), and $v_1(a) < 0$.

For any $\gamma > v_0(a)$, we have $x_{0,a}(t) = \mathrm{O}(\mathrm{e}^{\gamma t})$, $t \in \mathbb{R}_+$. In particular, $(x_{0,a}(t))_{t\in\mathbb{R}_+}$ is square integrable if and only if, (see Gushchin and Küchler (2000)) $v_0(a) < 0$. The Laplace transform of $(x_{0,a}(t))_{t\in\mathbb{R}_+}$ is given by

$$\int_0^\infty \mathrm{e}^{-\lambda t} x_{0,a}(t)\,\mathrm{d}t = \frac{1}{h_a(\lambda)}, \quad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) > v_0(a).$$

Based on the inverse Laplace transform and Cauchy's residue theorem, the following crucial lemma can be shown (see, e.g., Gushchin and Küchler (1999, Lemma 1.1)).