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Bias-correction and endogenous lag order algorithm for bootstrap prediction intervals

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ABSTRACT

Pan and Politis present an informative and comprehensive review of the bootstrap methods for constructing prediction intervals for autoregressive (*AR*) time series. In this discussion, I call attention to the bias-correction and endogenous lag order algorithms, which can be added to the bootstrap procedures. They can be implemented with resampling based on the forward *AR* model, backward *AR* model, and predictive residuals. Using a long time series of price–earnings ratio, I demonstrate that these additional techniques substantially improve the small sample performance of the bootstrap prediction intervals for linear *AR* models.

1. Bias-correction

The *AR* parameter estimators are intrinsically biased in small samples, and this makes the bootstrap sample necessarily biased. For example, for the $AR(1)$ model $X_t = \phi_0 + \phi_1 X_{t-1} + \varepsilon_t$, the bias of least-squares (LS) estimator $\hat{\phi}_1$ for ϕ_1 to $O(n^{-1})$ is given by

$$E(\hat{\phi}_1) - \phi_1 = -\frac{1 + 3\phi_1}{n} + O(n^{-2}), \tag{1}$$

where n is the sample size. Based on the above (asymptotic) formula, the bias-corrected estimator for ϕ_1 unbiased to $O(n^{-1})$ is derived as

$$\hat{\phi}_1^{AC} = \frac{1 + n\hat{\phi}_1}{n - 3}, \tag{2}$$

where $E(\hat{\phi}_1{}^{AC}) = \phi_1 + O(n^{-2})$. For the linear $AR(p)$ model with $p > 1$, Kim (2004) derives a matrix formula for the bias-corrected parameter estimators for *AR* parameters unbiased to $O(n^{-1})$, based on the bias expressions given by Stine and Shaman (1989). Even if $\hat{\phi}_1{}^{AC}$ is used to generate a bootstrap sample, the bootstrap estimator $\hat{\phi}_1^*$ is still biased and should be corrected. That is, in each bootstrap replication, we employ

$$\hat{\phi}_1^{*AC} = \frac{1 + n\hat{\phi}_1^*}{n - 3} \tag{3}$$

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1 The R codes for the empirical application, using the R package BootPR (Kim, 2014), are available from the author on request.

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to generate bootstrap forecasts. Alternatively, the bias-corrected bootstrap can be conducted based on two successive bootstrap procedures (or bootstrap-after-bootstrap). That is, obtain the bias-corrected estimator

$$\hat{\phi}_1^{BC} = \hat{\phi}_1 - Bias(\hat{\phi}_1), \tag{4}$$

where $Bias(\hat{\phi}_1)$ denotes the bias estimator of $\hat{\phi}_1$ obtained from the first bootstrap. In the second bootstrap, bias-correction is made as

$$\hat{\phi}_1^{*BC} = \hat{\phi}_1^{*} - Bias(\hat{\phi}_1), \tag{5}$$

using $Bias(\hat{\phi}_1)$ as a proxy for $Bias(\hat{\phi}_1^*)$. The bias-corrected estimator (5) is employed in each bootstrap replication to generate bootstrap forecasts.

Kilian (1998a, 1998b) provides the full details of the above bias-corrected bootstrap methods in a more general setting. It is possible that bias-correction pushes the AR parameter estimates to non-stationarity. In this case, Kilian's (1998) stationary-correction is used, which shrinks the bias estimate until the bias-corrected estimate satisfies the condition of stationarity. Clements and Taylor (2001) and Kim (2001) propose bootstrap prediction intervals for the $AR(p)$ model based on bootstrap bias-correction ((4) and (5)): the former using the forward model for resampling, while the latter the backward one. Kim (2004) considers the case of asymptotic bias-correction ((2) and (3)) with resampling based on the backward model. Clements and Taylor (2001) and Kim (2001, 2004) provide Monte Carlo evidence that the bootstrap prediction intervals with bias-correction perform much better than those without, especially when the sample size is small and the model is near non-stationary.

2. Endogenous lag order algorithm

Consider the $AR(p)$ model of the form

$$X_t = \phi_0 + \beta t + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-1} + \varepsilon_t. \tag{6}$$

The sampling variability associated with the estimation of unknown AR order is an additional source of uncertainty to parameter estimation. The endogenous lag order algorithm treats the AR order for the bootstrap sample as unknown and estimates the order in each bootstrap replication (Masarotto, 1990; Kilian, 1998c). Let p^* be the AR order for $\{X_t\}_{t=1}^n$ estimated using an order selection criterion. We use an $AR(p^*)$ model to generate the bootstrap sample $\left\{X_t^*\right\}_{t=1}^n$, and again use an order selection criterion to estimate the AR order p^{**} of $\left\{X_t^*\right\}_{t=1}^n$. The bootstrap forecasts are generated using an $AR(p^{**})$ model. The bias-correction method based on asymptotic formula (as in (2) and (3)) can be implemented along with the endogenous lag order algorithm, due to its computational efficiency. Note that the effect of order selection is asymptotically negligible (see Kilian, 1998c). Kim (2004) provides the full details of the bias-corrected bootstrap with endogenous lag order algorithm for the AR model (6), with the Monte Carlo evidence that it provides the prediction interval with highly desirable small sample properties.

3. Empirical evaluation

As an application, I conduct interval forecasting of the PE (price–earnings) ratio for a US stock index (S&P Composite), monthly from Jan. 1881 to Dec. 2013 (1596 observations),[2] which is plotted in Fig. 1. The PE ratio is the price of stocks relative to the earnings, which a key indicator for stock valuation. The aim of this exercise is to evaluate the performance of alternative bootstrap prediction intervals in an empirical setting with real time series data. I take rolling-window of size 120. That is, I take the first window from Jan. 1881 to Dec. 1890 to fit the model (6) and generate h-step ahead (out-of-sample and dynamic) prediction intervals; and then take the second window from Feb. 1881 to Jan. 1891 again generating h-step ahead prediction intervals. This process continues to the end of the data set, generating a total of 1465 prediction intervals for all $h = 1, \ldots, 12$.

The unknown AR order in (6) is estimated (p^*) using Akaike's information criterion (AIC) with the maximum order set to 12. For the endogenous lag order algorithm, AIC is again used to estimate the AR order of bootstrap samples (p^{**}) with the maximum order set to 18. To evaluate the performance of prediction intervals, I use the coverage rate and mean interval score for all h, averaged over 1465 prediction intervals. Let X_{n+h} be the true future value; and $U(h)$ and $L(h)$ be the upper and lower limits of $100(1-\alpha)\%$ h-step ahead prediction interval. The interval score is defined as (Gneiting and Raftery, 2007; p. 370)

$$S(h) = (U(h) - L(h)) + \frac{2}{\alpha}(L(h) - X_{n+h})I(X_{n+h} < L(h)) + \frac{2}{\alpha}(X_{n+h} - U(h))I(X_{n+h} > U(h)),$$

where $I()$ is an indicator function. The coverage rate is the proportion of prediction intervals covering the true future values. The prediction intervals based on the Box–Jenkins method; bootstrap without bias-correction; bootstrap with asymptotic

[2] The data is compiled and made available by Robert Shiller (http://www.econ.yale.edu/~shiller/).