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# Bias-correction of Kalman filter estimators associated to a linear state space model with estimated parameters

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ABSTRACT

This paper aims to discuss some practical problems on linear state space models with estimated parameters. While the existing research focuses on the prediction mean square error of the Kalman filter estimators, this work presents some results on bias propagation into both one-step ahead and update estimators, namely, non recursive analytical expressions for them. In particular, it is discussed the impact of the bias in the invariant state space models. The theoretical results presented in this work provide an adaptive correction procedure based on any parameters estimation method (for instance, maximum likelihood or distribution-free estimators). This procedure is applied to two data set: in the calibration of radar precipitation estimates and in the global mean land–ocean temperature index modeling.



## 1. Introduction

State space models have been largely applied in several areas of applied statistics. In particular, the linear state space models have desirable properties and they have a huge potential in time series modeling that incorporates latent processes.

Once a model is placed in the linear state space form, the most usual algorithm to predict the latent process, the state, is the Kalman filter algorithm. This algorithm is a procedure for computing, at each time $t$ ($t = 1, 2, \ldots$), the optimal estimator of the state vector based on the available information until $t$ and its success lies on the fact that is an online estimation procedure. The main goal of the Kalman filter algorithm is to find predictions for the unobservable variables based on observable variables related to each other through a set of equations forming the state space model. Indeed, in the context of linear state space models, the Kalman filter produces the best linear unbiased estimators. When the errors and the initial state are Gaussian, the Kalman filter estimators are the best unbiased estimators in the sense of the minimum mean square error. However, the optimal properties only can be guaranteed when all model's parameters are known (Harvey, 1996). If the model is nonlinear, it must be considered the equation of optimal filtering (Stratonovich, 1960; Dobrovidov et al., 2012). However, as it was proved in Markovich (2015), when the unobservable Markov sequence is defined by a linear equation with a Gaussian noise, the equation of optimal filtering coincides with the classical Kalman filter.

In practice, some or even all model's parameters are unknown and have to be estimated. When the true parameters $\Theta$ of the linear state space model are, for instance, substituted by their maximum likelihood ML (or other) estimates, $\widehat{\Theta}$, the theoretical properties of Kalman filter estimators are no longer valid. The usual approach in the analysis of the effects (implications) of applying estimates rather than using true values is to recalculate the mean square errors of both one-step-ahead

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estimator and update estimator of the unknown state $\beta_t$, $P_{t|t-1}$ and $P_{t|t}$, respectively. This approach is discussed in the literature, for instance in Ansley and Kohn (1986) and Hamilton (1986) or more recently in Pfeffermann and Tiller (2005) and it relies on the fact that substituting the model parameters by their estimates in the theoretical mean square error (MSE) expression, that assumes known parameters values, results in underestimation of the true MSE.

Indeed, denoting by $\widehat{\beta}_{t|t}(\widehat{\Theta})$ the optimal filter estimator of $\beta_t$ based on the observations up to time $t$ substituting $\Theta$ by $\widehat{\Theta}$, the MSE of the estimation error is

$$\begin{aligned}\mathrm{MSE}_{t|t} &= \mathrm{E}\left\{\left[\widehat{\beta}_{t|t}(\widehat{\Theta}) - \beta_t\right]\left[\widehat{\beta}_{t|t}(\widehat{\Theta}) - \beta_t\right]'\right\}\\ &= P_{t|t} + \mathrm{E}\left\{\left[\widehat{\beta}_{t|t} - \widehat{\beta}_{t|t}(\widehat{\Theta})\right]\left[\widehat{\beta}_{t|t} - \widehat{\beta}_{t|t}(\widehat{\Theta})\right]'\right\}.\end{aligned}$$

The first term of the sum is the uncertainty contribution of the Kalman filter resulting from the estimation of state when the model parameters are known. The second term reflects the uncertainty due to the estimation of parameters.

Usually, the existent literature investigates methodologies to the second parcel, that is, the contribution to the $\mathrm{MSE}_{t|t}$ resulting from 'parameters uncertainty'. In Hamilton (1986) it is suggested the application of Monte Carlo techniques combining with the ML estimation. From another perspective, Ansley and Kohn (1986) proposed to approximate $P_{t|t}$ by $P_{t|t}(\widehat{\Theta})$ and to expand $\widehat{\beta}_{t|t}(\widehat{\Theta})$ around $\widehat{\beta}_{t|t}$ until the second term. These works were extended in a Bayesian approach in Quenneville and Singh (2000). Wall and Stoffer (2002) proposed a bootstrap procedure for evaluating conditional forecast errors that requires the backward representation of the model. Tsimikas and Ledolter (1994) presented an alternative way to build the restricted likelihood function, also using mixed effects models.

Pfeffermann and Tiller (2005) studied non-parametric and parametric bootstrap methods. Also, a bootstrap approach was adopted in the estimation of the mean squared prediction error of the best linear estimator of nonlinear functions of finitely many future observations in a stationary time series in Bandyopadhyay and Lahiri (2010). Rodríguez and Ruiz (2012) proposed two new bootstrap procedures to obtain MSE of the unobserved states which have better finite sample properties than both bootstraps alternatives and procedures based on the asymptotic approximation of the parameter distribution.

In this work it is investigated the parameters bias propagation into Kalman filter estimators, which allows proposing an adaptive correction algorithm of Kalman filter estimators bias based on an initial parameters estimates. This procedure allows an improvement in modeling of two relevant applications: the calibration of radar precipitation estimates and in the modeling of the global mean land–ocean temperature index between 1880 and 2013.

## 2. The state space model

Consider the linear state space model represented by the equations

$$Y_t = H_t\beta_t + e_t \tag{1}$$
$$\beta_t = \mu + \Phi(\beta_{t-1} - \mu) + \varepsilon_t, \tag{2}$$

where $Y_t$ is a $k \times 1$ vector time series of observable variables at time $t$, which are related with the $m \times 1$ vector of unobservable state variables, $\beta_t$, known as the state vector, $\mu$ is a $m \times 1$ vector of parameters, $\Phi$ is a $m \times m$ transition matrix and the disturbances $e_t$ and $\epsilon_t$ are $k \times 1$ and $m \times 1$ vectors, respectively, of serially uncorrelated white noise processes with zero mean and covariance matrices $\Sigma_e = E(e_t e_t')$, $\Sigma_\varepsilon = E(\varepsilon_t \varepsilon_t')$ and $E(e_t \epsilon_s') = \mathbf{0}$ for all $t$ and $s$. Although the state process $\{\beta_t\}$ is not observable, it is generated by a first-order autoregressive process according to (2), the *transition equation*. All the $k \times m$ matrices $H_t$ are assumed to be known at time $t-1$.

An important class of state space models is given by Gaussian linear state space models when the disturbances $e_t$ and $\varepsilon_t$ and the initial state are Gaussian. The state space model (1)–(2) does not impose any restriction on the stationarity of the state process $\{\beta_t\}$. However, in many applications there is no reason to assume that the state process is not stationary.

When the state process's stationarity is suitable it can be assumed that the state vector $\beta_t$ is a stationary VAR(1) process with mean $E(\beta_t) = \mu$ and transition matrix $\Phi$ with all eigenvalues inside the unit circle, i.e.,

$$|\lambda_i(\Phi)| < 1 \quad \text{for all } \lambda_i \text{ such that } |\Phi - \lambda_i I| = 0, \tag{3}$$

and with covariance matrix $\Sigma$, which is the solution of the equation $\Sigma = \Phi\Sigma\Phi' + \Sigma_\varepsilon$.

Usually, the linear state space models are represented considering a state equation as

$$\beta_t = \Phi\beta_{t-1} + \varepsilon_t$$

or in a simple way taking $\Phi = I$, i.e., considering that the state process $\{\beta_t\}$ is a random walk. However, the state space formulation (1)–(2) is more general since this formulation additionally allows the state to be a nonzero mean stationary process. When the state process $\{\beta_t\}$ is non-stationary the transition equation can be rewritten as $\beta_t = C + \Phi\beta_{t-1} + \varepsilon_t$, where $C = (I - \Phi)\mu$ and the state may be non-stationary VAR(1) process.