Statistical Methodology 31 (2016) 1–7

Contents lists available at ScienceDirect

Statistical Methodology

journal homepage: www.elsevier.com/locate/stamet

A note on domains of attraction of the limit laws of intermediate order statistics under power normalization

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ARTICLE INFO

Article history:
Received 9 September 2014
Received in revised form
2 December 2015
Accepted 9 January 2016
Available online 19 January 2016

MSC:
60G70
62E20

Keywords:
Extreme value theory
Intermediate order statistics
Domains of attraction
Power normalization

ABSTRACT

In this paper we compare the domains of attraction of limit laws of intermediate order statistics under power normalization with those of limit laws of intermediate order statistics under linear normalization. As a result of this comparison, we obtain necessary and sufficient conditions for a univariate distribution function to belong to the domain of attraction for each of the possible limit laws of intermediate order statistics under power normalization.

1. Introduction

Let $\{X_n\}$ be a sequence of independent random variables (rv's) with common continuous distribution function F and $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics from a random sample of size n. The term $X_{k_n:n}$ is called a left (or right) intermediate order statistic, if its rank sequence $\{k_n\}$ is such that $\min(k_n, n - k_n) \to \infty$ and $\frac{k_n}{n} \to 0$ (or $\frac{k_n}{n} \to 1$), as $n \to \infty$.

The intermediate order statistics have many applications. For example, the intermediate order statistics can be used to estimate the probabilities of future extreme observations and the tail quantiles

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http://dx.doi.org/10.1016/j.stamet.2016.01.001
1572-3127/

of the underlying distribution that are extremes relative to the available sample size. Pickands [12] has shown that intermediate order statistics can be used in constructing consistent estimators for the shape parameter of the limiting extremal distribution in the parametric form. Many authors, e.g., [15,9], have also found estimators that are based, in part, on intermediate order statistics.

Chibisov [5] and Wu [16] both have shown that the normal and lognormal distributions are possible limiting distributions for the intermediate order statistics. Since we can translate the result obtained for left order statistics into the case of right order statistics and vice versa, we consider only the left intermediate order statistics and begin with a basic theorem of Chibisov [5], in which the class of all possible weak limits for lower intermediate order statistics is given, by using the extreme value theory (see, [8]).

Theorem 1.1. *Let the rank sequence $\{k_n\}$ (which for abbreviation, sometimes is written as $\{k\}$) satisfy Chibisov's condition*

$$\lim_{n\to\infty}(\sqrt{k_{n+z_n(\nu)}}-\sqrt{k_n})=\frac{\alpha\ell}{2}\nu, \tag{1.1}$$

for every sequence of integers $z_n(\nu)$ such that $\frac{z_n(\nu)}{n^{1-\frac{\alpha}{2}}}\to\nu$, as $n\to\infty$, where $0<\alpha<1$, ν and $\ell>0$ are constants. Furthermore, let $a_n>0$ and $b_n\in\Re$ be normalizing constants for which the distribution function $F_{k:n}(a_nx+b_n)=P(X_{k:n}\leq a_nx+b_n)=I_{F(a_nx+b_n)}(k,n-k+1)$ weakly converges to a nondegenerate distribution function G, where $I_x(a,b)=\frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^x t^{a-1}(1-t)^{b-1}dt$, $0\leq x\leq 1$, $a,b\geq 1$, is the incomplete beta function. Then the limiting distribution G must be one and only one of the following types:

$$\begin{aligned} &G_{1,\beta}(x)=\Phi(v_{1,\beta}(x))=\Phi(\beta\log x),\quad x>0,\ \beta>0;\\ &G_{2,\beta}(x)=\Phi(v_{2,\beta}(x))=\Phi(-\beta\log|x|),\quad x\leq 0,\ \beta>0;\\ &G_3(x)=\Phi(v_3(x))=\Phi(x), \end{aligned} \tag{1.2}$$

where the function Φ is the standard normal distribution.

Clearly, the condition (1.1) implies the more applicable condition $\frac{k_n}{n^\alpha}\to\ell^2$, as $n\to\infty$. We can also show that the latter condition implies Chibisov's condition. Indeed, first we note that

$$\lim_{n\to\infty}\left((n+z_n(\nu))^{\frac{\alpha}{2}}-(n+\nu n^{1-\frac{\alpha}{2}})^{\frac{\alpha}{2}}\right)=\lim_{n\to\infty}\left(\frac{\alpha}{2}\times\frac{z_n(\nu)}{n^{1-\frac{\alpha}{2}}}(1+\circ(1))-\frac{\alpha}{2}\nu(1+\circ(1))\right)=0.$$

Therefore, we get

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{k_{n+z_n(\nu)}}-\sqrt{k_n}\right)&=\ell\lim_{n\to\infty}\left((n+z_n(\nu))^{\frac{\alpha}{2}}-n^{\frac{\alpha}{2}}\right)=\ell\lim_{n\to\infty}\left(\left(n+\nu n^{1-\frac{\alpha}{2}}\right)^{\frac{\alpha}{2}}-n^{\frac{\alpha}{2}}\right)\\ &=\ell\lim_{n\to\infty}\frac{\left(1+\nu n^{-\frac{\alpha}{2}}\right)^{\frac{\alpha}{2}}-1}{n^{-\frac{\alpha}{2}}}=\ell\lim_{m\downarrow 0}\frac{\left(1+\nu m^{\frac{\alpha}{2}}\right)^{\frac{\alpha}{2}}-1}{m^{\frac{\alpha}{2}}}.\end{aligned}$$

Thus, by applying L'Hopital's rule, Chibisov's condition immediately follows.

The above implication shows that the class of intermediate rank sequences which satisfy Chibisov's condition is a very wide class, and consequently, Theorem 1.1 is widely applicable. Later, Wu [16] generalized the result of Chibisov for any nondecreasing intermediate rank sequence and proved that the only possible types for the limit G are those defined in (1.2). Some complements to Wu [16] are done by Barakat and Ramachandran [4]. The domains of attraction of the limiting forms $G_{1,\beta}$, $G_{2,\beta}$ and G_3 have been obtained in [5]. The power normalization, $T_n(x)=c_n\mid x\mid^{d_n}\mathrm{sign}(x)$, $c_n,d_n>0$, was initially utilized by Pancheva [11] to derive a more accurate approximation of the distribution function of the maximum order statistics. Pancheva [11] has derived the six possible limit types of distribution functions of the power normalized maxima, $F^n(T_n(x))$, as $n\to\infty$, and Mohan and Ravi [10] called them p-max stable laws. Two distribution functions Ψ_1 and Ψ_2 are said to be of the same power type (p-type) if for some $A,B>0$, $\Psi_1(x)=\Psi_2(A|x|^B\mathrm{sign}(x))$, for all $x\in\mathbb{R}$. The class of p-max stable