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A random matrix from a stochastic heat equation[☆]

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ABSTRACT

We find a random matrix to study a stochastic heat equation (SHE), and in doing so, we propose a method to discretize stochastic partial differential equations. Moreover, the convergence result helps to corroborate that standard partitions in the deterministic problem can also be considered in the stochastic case. In our study, we focus on the stochastic Schrödinger operator associated to the SHE, and prove a weak convergence of the random matrix to the stochastic operator. We do this by defining properly the space where the operators act, and by constructing a proper projection using the matrix.

1. Stochastic heat model

When dealing with partial differential equations, numerical procedures help to approximate solutions. In particular it is well known that some Jacobi matrices help to study heat equations; such matrices arise from partitions of space and time into intervals, which give rise to a lattice. It is also known that behind the numerical scheme one has a discrete operator (the matrix) approximating a continuum one.

In this paper, we study a stochastic partial differential equation, namely the stochastic heat equation (SHE), and propose a discretization of it to obtain a numerical scheme based on a matrix, which is random due to the presence of the white noise. The very first issue is to see in which sense the random matrix converges to the stochastic operator associated to the SHE. We prove a result in this direction, by properly specifying the sequence of operators and the domain where they act.

More precisely, we focus on the SPDE:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + uw', \quad t > 0,\ x \in [0, 1] \tag{1}$$

where w' represents Gaussian space–time noise.

To study previous problem, one can first study the following one-dimensional operator for functions $u : [0, 1] \to \mathbb{R}$

$$Lu := \beta \frac{d^2 u}{dx^2} + u \times b', \quad x \in [0, 1], \tag{2}$$

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where b' is a Gaussian white noise on the interval $[0, 1]$. This is an instance of the so-called stochastic Schrödinger operator, and it is useful to recognize that it defines a weak stochastic operator in the sense of Skorohod (1984).

It turns out that the following $n \times n$ tridiagonal random matrix helps to study (or approximate) the previous random operator, $A_n :=$

$$\begin{bmatrix} \sqrt{n+1}\xi_1 - 2\beta(n+1)^2 & \beta(n+1)^2 & & \\ \beta(n+1)^2 & \sqrt{n+1}\xi_2 - 2\beta(n+1)^2 & \beta(n+1)^2 & \\ & & \ddots & \\ & & \beta(n+1)^2 & \sqrt{n+1}\xi_n - 2\beta(n+1)^2 \end{bmatrix}, \tag{3}$$

where $\xi_1, \dots, \xi_n$ are i.i.d. $N(0, 1)$ r.v.s.

It is important to take to following point of view. We can see the sequence $A_1, A_2, \dots$ as random operators acting on the same domain of the operator L; then we can try to see in which sense A_n converges to L. It turns out that $A_n \to L$ weakly in the sense of Skorohod (1984).

When working with numerical schemes for deterministic partial differential equations, it is known that the correct relation in the space–time partition is

$$\alpha\Delta x = \sqrt{\Delta t},$$

for some positive constant α. As we will mention in Remark 3, we will be able to confirm that this relation also makes sense for analyzing numerically the SHE.

Before we get started, we must mention that there is already a theory for numerical methods to study SPDEs (see e.g. Walsh, 2005), but the present work goes in a different perspective.

2. The deterministic case

Suppose we want to solve the following boundary value problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in [0, 1] \\ &u(0, t) = u(1, t) = 0, \quad t > 0, \\ &u(x, 0) = f(x), \quad x \in [0, 1], \end{aligned} \tag{4}$$

where f is such that the problem has a unique solution.

A typical numerical scheme reduces the problem to a finite dimensional one (see for instance Elaydi, 2005 and Kress, 1998). Let $\{t_0 := 0, t_1, \dots\}$ and

$$\left\{x_0 := 0, x_1 = \frac{1}{n+1}, \dots, x_n := \frac{n}{n+1}, x_{n+1} := 1\right\}$$

be equidistant partitions of the time and the space, respectively.

Let $T_i(k) := u(x_i, t_k)$, with $T_0(k) = T_{n+1}(k) = 0$. Then the discrete counterpart of (4) is

$$\frac{T_i(k+1) - T_i(k)}{\Delta t} = \beta\frac{T_{i+1}(k) - 2T_i(k) + T_{i-1}(k)}{(\Delta x)^2}, \quad 1 \le i \le n-1,$$

where $\Delta t := t_1 - t_0$, $\Delta x := x_1 - x_0 = \frac{1}{n+1}$. By defining $T(k) := (T_1(k), \dots, T_n(k))^T$ we can write

$$T(k+1) = \left(I - \beta\frac{\Delta t}{(\Delta x)^2}A\right)T(k), \quad k = 1, 2, \dots,$$

with the $n \times n$ matrix

$$A := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & -1 & 2 \end{bmatrix}.$$

Remark 1. When dealing with the numerical schemes described above one can impose some condition on Δt and Δx in order to achieve the so-called stability. Roughly speaking this means that the possible error, aroused at first stages of the numerical procedure, does not get magnified as the procedure carries on; in other words, the error remains bounded. Such condition (see e.g. Smith, 1985) is given by

$$2\beta\frac{\Delta t}{(\Delta x)^2} \le 1. \tag{5}$$

Thus, if we set $\Delta x := \frac{1}{n+1}$ it suffices to take Δt proportional to $\frac{1}{(n+1)^2}$.