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A characterization of the generalized Laplace distribution by constant regression on the sample mean

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ABSTRACT

This note introduces the generalized Laplace distribution having two scale parameters and one location parameter (for which the ordinary Laplace distribution is a special case). This distribution is then characterized by a constant regression of a certain polynomial statistic on the sample mean in the sense of such characterizations initiated by Laha and Lukacs (1960).

1. Introduction

Since the seminal work of Laha and Lukacs (1960), Lukacs and Laha (1964) on characterizations of distributions by the requirement of a constant (or zero) regression of a quadratic statistic on the sample mean, numerous and various such characterizations have been developed. A comprehensive study on the subject is the monograph by Kagan et al. (1973).

Bar-Lev (2007) introduced a comprehensive guideline for constructing such characterizations and applied them to natural exponential families (NEF's) which include most of the well known distributions characterized as such. His paper also contains an extensive list of citations (old and relatively recent) of works on the subject and the reader is referred to the references cited therein.

In this note we characterize the generalized Laplace distribution (which includes the ordinary Laplace distribution as a special case) by a constant (or zero) regression of some sample functionals on the sample mean.

Section 2 is devoted to the description of the generalized Laplace distribution (GLD) and some of its properties. The characterization theorem of the GLD is presented in Section 3.

2. The generalized Laplace distribution

The GLD, supported on $\mathbb{R}$, is parameterized by three parameters: two are scale parameters α and β and one is a location parameter θ, and is given by the p.d.f.

$$p(x;\alpha,\beta,\theta)=\frac{\alpha\beta}{\alpha+\beta}\left[e^{-\alpha|x-\theta|}I_{(x<\theta)}+e^{-\beta|x-\theta|}I_{(x\geq\theta)}\right],\quad \alpha>0,\ \beta>0,\ \theta\in\mathbb{R}, \tag{1}$$

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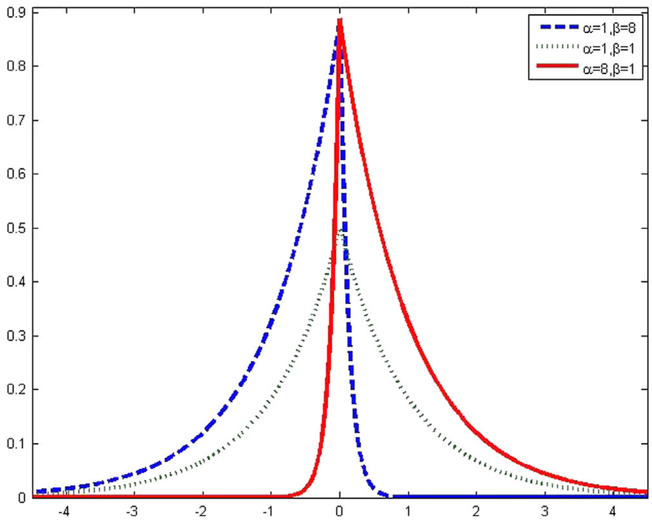

Fig. 1. Plots of $p(x; \alpha, \beta, 0)$ for $(\alpha, \beta) = (1, 8), (1, 1), (8, 1)$.

where I_A is the indicator function of a set A. If $\alpha = \beta$ then (1) reduces to the ordinary Laplace p.d.f.

$$p(x; \alpha, \alpha, \theta) = \frac{\alpha}{2} e^{-\alpha|x-\theta|} I_{(-\infty<x<\infty)}, \quad \alpha > 0,\ \theta \in \mathbb{R}. \tag{2}$$

Note that neither (1) or (2) constitutes an NEF or a general exponential family.

The generalized (or asymmetric) Laplace distribution was discussed by Kotz et al. (2001, Chapter 3) with respect to various statistical properties. It was also shown by Jorgensen and Kokonendji (2011) (see also Jorgensen and Kokonendji, 2016) to play a central role in the convergence of geometric dispersion models in a manner similar to that of Gaussian distribution in the convergence of classical dispersion models.

The moment generating function of (1) is

$$M(t; \alpha, \beta, \theta) = E(e^{tX}) = e^{t\theta} \alpha\beta \frac{1}{(\alpha + t)(\beta - t)}, \quad -\alpha < t < \beta,$$

and thus it is real analytic in neighborhood of 0. The corresponding characteristic function (c.f.) is therefore

$$f(t) = f(t; \alpha, \beta, \theta) = E(e^{itX}) = e^{it\theta} \alpha\beta \frac{1}{(\alpha + it)(\beta - it)}. \tag{3}$$

Let α_r and κ_r, $r \in \mathbb{N}$, denote, respectively, the rth moment and rth cumulant corresponding to (1). Then all such moments are finite and, as can easily be shown by simple induction, the rth cumulant has the form

$$\begin{aligned} \kappa_1 &= \theta + (\beta^{-1} - \alpha^{-1}) \\ \kappa_r &= (r-1)!(\beta^{-r} + (-1)^r \alpha^{-r}), \quad r \geq 2. \end{aligned} \tag{4}$$

Without loss of generality we henceforth assume that $\theta = 0$. Such an assumption does not affect the characterization of the GLD by constant regression on the sample mean (see a relevant remark at the end of the proof in Section 3), and we henceforth consider the p.d.f.

$$p(x; \alpha, \beta, 0) = \frac{\alpha\beta}{\alpha + \beta} \left[e^{-\alpha|x|} I_{(x<0)} + e^{-\beta|x|} I_{(x\geq 0)} \right], \quad \alpha > 0,\ \beta > 0, \tag{5}$$

and its c.f.

$$f(t) = f(t; \alpha, \beta, 0) = E(e^{itX}) = \alpha\beta \frac{1}{(\alpha + it)(\beta - it)} \tag{6}$$

for any further developments.

The p.d.f. $p(x; \alpha, \beta, 0)$ is unimodal at $x = 0$ with value $\alpha\beta(\alpha + \beta)^{-1}$. Fig. 1 displays this p.d.f. for various three pairs of (α, β): for $\alpha < \beta$ with $(\alpha = 1, \beta = 8)$; for $\alpha = \beta$ with $(\alpha = \beta = 1)$, i.e., the symmetric Laplace distribution; and for $(\alpha > \beta)$ with $(\alpha = 8, \beta = 1)$. It can readily be verified in general that if $\alpha > \beta$, the β-exponential tail is thicker than that of the α-exponential tail; and vice versa, if $\alpha < \beta$, then the α-exponential tail is thicker than that of the β-exponential tail.

Though additional statistical and probabilistic properties of the GLD are desirable, it is, however, beyond the scope of the present characterization study.