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Conditional independence among max-stable laws

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ABSTRACT

Let X be a max-stable random vector with positive continuous density. It is proved that the conditional independence of any collection of disjoint subvectors of X given the remaining components implies their joint independence. We conclude that a broad class of tractable max-stable models cannot exhibit an interesting Markov structure.

1. Introduction

As pointed out by Dawid (1979) *independence* and *conditional independence* are key concepts in the theory of probability and statistical inference. A collection of (not necessarily real-valued) random variables $Y_1, \dots, Y_k$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ are called conditionally independent given the random variable Z (on the same probability space) if

$$\mathbb{P}(Y_1 \in A_1, \dots, Y_k \in A_k \mid Z) = \prod_{i=1}^{k} \mathbb{P}(Y_i \in A_i \mid Z) \quad \mathbb{P}\text{-a.s.},$$

for any measurable sets $A_1, \dots, A_k$ from the respective state spaces. The conditioning is meant with respect to the σ-algebra generated by Z. A particularly important example for the conditional independence to be an omnipresent attribute are the *Gaussian Markov random fields* that have evolved as a useful tool in spatial statistics (Rue and Held, 2005; Lauritzen, 1996). Here, the zeros of the *precision matrix* (the inverse of the covariance matrix) of a Gaussian random vector represent precisely the conditional independence of the respective components conditioned on the remaining components of the random vector. Hence, sparse precision matrices are desirable for statistical inference.

In the analysis of the extreme values of a distribution (rather than fluctuations around mean values) *max-stable* models have been frequently considered. We refer to Engelke et al. (2014), Naveau et al. (2009), Blanchet and Davison (2011) and Buishand et al. (2008) for some spatial applications among many others. Their popularity originates from the fact that max-stable distributions arise precisely as possible limits of location-scale normalizations of i.i.d. random elements. A random

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vector X is called max-stable if it satisfies the distributional equality $a_nX + b_n \overset{\mathcal{D}}{=} \max(X^{(1)}, \dots, X^{(n)})$ for independent copies $X^{(1)}, \dots, X^{(n)}$ of X for some appropriate normalizing sequences $a_n > 0$ and $b_n \in \mathbb{R}$, where all operations are meant componentwise. If the components X_i of X are *standard Fréchet* distributed, i.e. $\mathbb{P}(X_i \le x) = \exp(-1/x)$ for $x \in (0, \infty)$, we have $a_n = n$ and $b_n = 0$ and the random vector X will be called *simple max-stable*.

Let I be a non-empty finite set. It is well-known (cf. e.g. Resnick, 2008) that the distribution functions G of simple max-stable random vectors $X = (X_i)_{i\in I}$ are in a one-to-one correspondence with Radon measures H on some reference sphere $S_+ = \{\omega \in [0, \infty)^I : \|\omega\| = 1\}$ that satisfy the moment conditions $\int \omega_i H(\mathrm{d}\omega) = 1$, $i \in I$. The correspondence between G and H is given by the relation

$$G(x) = \mathbb{P}(X_i \le x_i,\ i \in I) = \exp\left(-\int_{S_+} \max_{i\in I} \frac{\omega_i}{x_i} H(\mathrm{d}\omega)\right), \quad x \in (0, \infty)^I.$$

Here, $\|\cdot\|$ can be any norm on $\mathbb{R}^I$ and H is often called *angular* or *spectral measure*.

In general, neither does independence imply conditional independence nor does conditional independence imply independence of the subvectors of a random vector. Consider the following two simple examples which illustrate this fact in the case of Gaussian random vectors (Example 1) and max-stable random vectors (Example 2). For notational convenience, we write $X \perp\!\!\!\perp Y$ if X and Y are independent and $X \perp\!\!\!\perp Y \mid Z$ if X and Y are conditionally independent given Z and likewise use the instructive notation $\perp\!\!\!\perp_{i=1}^k X_i$ and $\perp\!\!\!\perp_{i=1}^k X_i \mid Z$ if more than two random elements are involved.

Example 1. Let X_1, X_2, X_3 be three independent standard normal random variables and, moreover, $X_4 = X_1 + X_2$ and $X_5 = X_1 + X_2 + X_3$. Then all subvectors of $(X_i)_{i=1}^5$ are Gaussian and

$$X_1 \perp\!\!\!\perp X_2, \qquad \text{but not} \qquad X_1 \perp\!\!\!\perp X_2 \mid X_5, \tag{1}$$

$$\text{whereas} \qquad X_1 \perp\!\!\!\perp X_5 \mid X_4, \qquad \text{but not} \qquad X_1 \perp\!\!\!\perp X_5. \tag{2}$$

Example 2. Let X_1, X_2, X_3 be three independent standard Fréchet random variables and, moreover, $X_4 = \max(X_1, X_2)$ and $X_5 = \max(X_1, X_2, X_3)$. Then all subvectors of $(X_i)_{i=1,\dots,5}$ are max-stable and both relations (1) and (2) hold true also in this setting.

However, if the distribution of a max-stable random vector has a positive continuous density, then conditional independence of any two subvectors conditioned on the remaining components implies already their independence. To be precise, when we say that a random vector has a *positive continuous density*, we mean that the joint distribution of its components has a positive continuous density. The following theorem is the main result of the present article. If $X = (X_i)_{i\in I}$ is a random vector, we write X_A for the subvector $(X_i)_{i\in A}$ if $A \subset I$. The same convention applies to non-random vectors $x = (x_i)_{i\in I}$.

Theorem 1. *Let $X = (X_i)_{i\in I}$ be a simple max-stable random vector with positive continuous density. Then, for any disjoint non-empty subsets A and B of I, the conditional independence $X_A \perp\!\!\!\perp X_B \mid X_{I\setminus(A\cup B)}$ implies the independence $X_A \perp\!\!\!\perp X_B$.*

A proof of this theorem is given in Section 3. Beforehand, some comments are in order.

(a) Firstly, the requirement of a positive continuous density for X is much less restrictive than requiring the spectral measure H of X to admit such a density, cf. Beirlant et al. (2004, pp. 262–264) and references therein. For instance, fully independent variables $X = (X_i)_{i\in I}$ have a discrete spectral measure, while their density exists and is positive and continuous. A more subtle example is, for instance, the asymmetric logistic model (Tawn, 1990), which admits a continuous positive density and whose spectral measure carries mass on all faces of S_+, cf. also Example 3.

(b) Secondly, both random vectors $(X_i)_{i=1,2,5}$ and $(X_i)_{i=1,4,5}$ that were considered in the Gaussian case in Example 1 have a positive continuous density on $\mathbb{R}^d$. Hence, there exists no version for Theorem 1 for the Gaussian case.

(c) Note that the implication of Theorem 1 is the independence of X_A and X_B, not the independence of all three subvectors $X_A, X_B, X_{I\setminus(A\cup B)}$.

(d) By means of the same argument that shows that pairwise independence of the components of a max-stable random vector implies already their joint independence, we may deduce a version of Theorem 1, in which more than two subvectors are considered.

Corollary 2. *Let $X = (X_i)_{i\in I}$ be a simple max-stable random vector with positive continuous density. Then, for any disjoint non-empty subsets $A_1, \dots, A_k$ of I, the conditional independence $\perp\!\!\!\perp_{i=1}^k X_{A_i} \mid X_{I\setminus\bigcup_{i=1}^k A_i}$ implies the independence $\perp\!\!\!\perp_{i=1}^k X_{A_i}$.*

(e) The non-degenerate univariate max-stable laws are classified up to location and scale by the one parameter family of extreme value distributions indexed by $\gamma \in \mathbb{R}$

$$F_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}), \quad x \in \begin{cases} (-1/\gamma, \infty) & \gamma > 0, \\ \mathbb{R} & \gamma = 0, \\ (-\infty, -1/\gamma) & \gamma < 0. \end{cases}$$