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Intrinsic random functions and universal kriging on the circle

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ABSTRACT

Intrinsic random functions (IRF) provide a versatile approach when the assumption of second-order stationarity is not met. Here, we develop the IRF theory on the circle with its universal kriging application. Unlike IRF in Euclidean spaces, where differential operations are used to achieve stationarity, our result shows that low-frequency truncation of the Fourier series representation of the IRF is required for such processes on the circle. All of these features and developments are presented through the theory of reproducing kernel Hilbert space. In addition, the connection between kriging and splines is also established, demonstrating their equivalence on the circle.

1. Introduction

When a random process is considered on a circle, it is often assumed to be second-order stationary (or stationary for short in the paper), that is, the mean of the process is constant over the circle and the covariance function at any two points depends only on their angular distance (Yaglom, 1961; Roy, 1972; Roy and Dufour, 1974; Dufour and Roy, 1976; Wood, 1995; Gneiting, 1998). While stationarity is commonly assumed, it is often considered to be unrealistic in practice. In Euclidean spaces, a large class of non-stationary phenomena may be represented through intrinsic random functions (IRF, Matheron, 1973; Cressie, 1993; Chilès and Delfiner, 2012). The properties of IRF in other spaces, such as the circle or sphere, are not widely known. In this paper, the theory of IRF on the circle is developed, where we find that instead of differential operations, truncation of the Fourier series representation becomes essential for IRF on the circle. This can be presented in the context of the reproducing kernel Hilbert space (RKHS, Aronszajn, 1950; Wahba, 1990a). We formally make such a connection and further relate universal kriging with the smoothing formula in RKHS. Based on this approach, we are able to demonstrate the equivalence between splines and kriging on the circle.

2. IRF and RKHS

A key component for IRF is the allowable measure. Based on Matheron (1973) and Chilès and Delfiner (2012, Chapter 4), a discrete measure $\lambda = \sum_{i=1}^{m} \lambda_i \delta(t_i)$ on a unit circle S, where $t_i \in S$, $\lambda_i \in \mathbb{R}$ and $\delta(\cdot)$ is the Dirac measure, is allowable at the order of an integer $\kappa\,(\kappa \geq 0)$ if it annihilates all trigonometric functions of order $k < \kappa$. That is,

$$\sum_{i=1}^{m} \lambda_i \cos(kt_i) = \sum_{i=1}^{m} \lambda_i \sin(kt_i) = 0, \quad 0 \leq k < \kappa. \tag{1}$$

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We call Λ_κ the class of such allowable measures. Clearly $\Lambda_{\kappa+1} \subset \Lambda_\kappa$. In addition, for $\lambda \in \Lambda_\kappa$, the translated measure $\iota_t\lambda = \sum_{i=1}^m \lambda_i\delta(t_i + t), t \in S$ remains in Λ_κ. This can be easily seen from the elementary trigonometric identities (also see Matheron, 1979; Chilès and Delfiner, 2012). For any function $f(\cdot)$ on S, we define $f(\lambda) = \sum_{i=1}^m \lambda_i f(t_i)$.

In this paper, we consider a random process $\{Z(t), t \in S\}$ on a unit circle with finite second moment and being continuous in quadratic mean. By Yaglom (1961) and Roy (1972), the process can be expanded in a Fourier series which is convergent in quadratic mean:

$$Z(t) = Z_0 + \sum_{n=1}^{\infty}(Z_{n,c}\cos nt + Z_{n,s}\sin nt), \tag{2}$$

where $Z_0 = 1/(2\pi)\int_S Z(t)dt$, $Z_{n,c} = (1/\pi)\int_S Z(t)\cos ntdt$, and $Z_{n,s} = (1/\pi)\int_S Z(t)\sin ntdt$.

Definition 2.1. For an integer $\kappa(\kappa \geq 0)$, the random process in (2) is called an IRFκ if for any $\lambda \in \Lambda_\kappa$, the process

$$Z_\lambda(t) = Z(\iota_t\lambda) = \sum_{i=1}^m \lambda_i Z(t_i + t)$$

is stationary with respect to $t \in S$ and has a zero mean.

To characterize such a circular IRFκ, we denote

$$Z_\kappa(t) = \sum_{n=\kappa}^{\infty}(Z_{n,c}\cos nt + Z_{n,s}\sin nt),$$

as its low-frequency truncated process and so we have the following lemma.

Lemma 2.1. *A random process given by (2) is an IRFκ if and only if its low-frequency truncated process $Z_\kappa(t)$ is stationary and has a zero mean.*

Proof. In the Fourier expansion (2), the lower trigonometric functions will be annihilated by $\lambda \in \Lambda_\kappa$, which implies

$$Z(\iota_t\lambda) = \sum_{n=\kappa}^{\infty}\left(Z_{n,c}\sum_{i=1}^m(\lambda_i\cos nt_i\cos nt - \lambda_i\sin nt_i\sin nt) + Z_{n,s}\sum_{i=1}^m(\lambda_i\sin nt_i\cos nt + \lambda_i\cos nt_i\sin nt)\right).$$

Denote $\lambda_{n,c} = \sum_{i=1}^m \lambda_i\cos nt_i$, $\lambda_{n,s} = \sum_{i=1}^m \lambda_i\sin nt_i$ and

$$Y_{n,c} = Z_{n,c}\lambda_{n,c} + Z_{n,s}\lambda_{n,s}, \qquad Y_{n,s} = -Z_{n,c}\lambda_{n,s} + Z_{n,s}\lambda_{n,c}, \quad n = \kappa, \kappa+1, \ldots, \tag{3}$$

we have

$$Z(\iota_t\lambda) = \sum_{n=\kappa}^{\infty}(Y_{n,c}\cos nt + Y_{n,s}\sin nt).$$

Yaglom (1961, Theorem 5) shows that a random process (2) on the circle is stationary if and only if its Fourier coefficients are uncorrelated random variables. The lemma can be directly obtained based on this and the linear mapping between the coefficients of $(Z_{n,c}, Z_{n,s})$ and $(Y_{n,c}, Y_{n,s})$ in (3). □

Remark 2.1. In Euclidean spaces, the IRF is associated with differential operations (Matheron, 1973; Chilès and Delfiner, 2012). For example, a differentiable IRFκ on a real line is characterized as that its $(\kappa+1)$ derivative is stationary. Lemma 2.1 indicates that for circular processes, the low-frequency truncation operation replaces differential operations and leads to stationarity. This observation also has important implications for splines on the circle, which is addressed in Section 4.

Remark 2.2. It is clear that an IRF0 on the circle is the conventional stationary process. Note that this is slightly different from what has been defined in the Euclidean spaces, where IRF(-1) is usually a stationary process. For the rest of this paper, we assume $\kappa \geq 1$ for notational simplicity.

Remark 2.3. Based on Lemma 2.1, for an IRFκ process $Z(t)$, the random process

$$Z^*(t) = Z(t) + A_0 + \sum_{n=1}^{\kappa-1}(A_{n,c}\cos nt + A_{n,s}\sin nt),$$

where $A_0, A_{n,c}, A_{n,s}, n = 1, \ldots, (\kappa-1)$ are random variables, is clearly also an IRFκ. These two processes $Z(t)$ and $Z^*(t)$ share the same truncation process $Z_\kappa(t)$, with $Z^*(\lambda) = Z(\lambda)$, for any $\lambda \in \Lambda_\kappa$. Similar to the discussion in Chilès and Delfiner (2012, Section 4.4.2), these functions form an equivalent class.