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A note on the maximal outdegrees of Galton–Watson trees

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ABSTRACT

In this note we consider both the local maximal outdegrees and the global maximal outdegree of Galton–Watson trees. In particular, we show that the tail of any local maximal outdegree and that of the offspring distribution are asymptotically equivalent up to a certain factor. However for the global maximal outdegree, this is only true in the subcritical case.

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1. Introduction

In a Galton–Watson tree (GW tree), we call the number of offsprings of a vertex the *outdegree* of the vertex. Then naturally by the *maximal outdegree* of a GW tree we mean the maximal number of offsprings of all the vertices in the tree. Recently, Bertoin (2011, 2013) has studied the maximal outdegree of GW trees in two special critical cases and its scaling limit. In particular, when the offspring distribution has a regularly varying tail, the explicit asymptotic of the tail of the maximal outdegree was given in Bertoin (2011, 2013). Also very recently, Kortchemski (2015) has given some uniform sub-exponential tail bounds for the width, height and maximal outdegree of critical GW trees conditioned on having a large fixed size, again assuming that the offspring distribution has a regularly varying tail. More generally about extremes in branching processes, see the surveys by Yanev (2007, 2008) and the references therein for a variety of existing results.

In this note, we consider both the *local maximal outdegrees* and the *global maximal outdegree* of GW trees and study their tails. A local maximal outdegree of a GW tree is the maximal number of offsprings of some but not all vertices in the tree. We use M_n to denote the maximal number of offsprings of all vertices at generation n, and $M_{[0,n]}$ the maximal number of offsprings of all vertices in the first $n+1$ generations, including generation 0. Clearly M_n and $M_{[0,n]}$ are some local maximal outdegrees. We also use $M = M_{[0,\infty)}$ to denote the supremum of all outdegrees in the tree, which is called the global maximal outdegree. In this note we show that the tail of M_n or $M_{[0,n]}$ is always asymptotically equivalent to the tail of the offspring distribution, up to a certain factor, and the tail of M is asymptotically equivalent to the tail of the offspring distribution, up to a certain factor, if and only if the offspring distribution is subcritical.

Now let us give a review of our results. Lemma 2.1 gives an expression of the finite-dimensional distribution of M_n at different generations. For M_n, Proposition 2.2 shows that its tail and that of the offspring distribution are asymptotically equivalent up to a certain factor. For $M_{[0,n]}$, a similar result is given in Theorem 3.1. A generalization of both Proposition 2.2 and Theorem 3.1 is given in Theorem 3.2. For M, we prove in Theorem 4.2 that in the subcritical case, the tail of the offspring distribution and that of M are asymptotically equivalent up to a factor $1-\mu$, where μ is the expectation of the offspring

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distribution. This asymptotical equivalence does not hold in the critical or supercritical case, however see Theorem 4.4 for some results on the tail of M in the critical case.

This note has been motivated by Bertoin (2011, 2013), He (2014) and He and Li (2014). Actually our Theorem 4.2 in this note was first proved in He (2014), by a barehands method. In this note, we prove all the results including Theorem 4.2, by a calculus method based on the mean value theorem and the Taylor's theorem. We feel that this calculus method is more revealing and more efficient. The topic of He and Li (2014) is the maximal jumps of continuous-state branching processes, which correspond to the maximal outdegrees of GW trees. In particular, it has been shown in He and Li (2014) that the tail of any local maximal jump and that of the Lévy measure are asymptotically equivalent up to a certain factor, and for the global maximal jump, this is only true in the subcritical case. So our main results in this note may be regarded as discrete analogues of those results in He and Li (2014).

This note is organized as follows. In Section 2, we review some notations of GW trees and prove two auxiliary results. Then we study the local maximal outdegrees in Section 3, and the global maximal outdegree in Section 4.

2. Auxiliary results

In this section we first review some notations of GW trees, then prove two auxiliary results.

We use $\tau = \tau(p)$ to denote a Galton–Watson tree (GW tree) with the *offspring distribution* $p = (p_0, p_1, p_2, \dots)$ on nonnegative integers. We say that p is *bounded* if the set $\{r; p_r > 0\}$ is bounded, and *unbounded* otherwise. Use $\mathbf{P}$ to denote the underlying probability, and μ the expectation of p. We assume that $0 < \mu < \infty$, so that $p_0 < 1$. We say that p is *(sub)critical* if it is critical or subcritical, that is, $\mu \le 1$. It is well-known that in the (sub)critical case a.s. $\tau(p)$ has finitely many vertices, and in the supercritical case with positive probability $\tau(p)$ has infinitely many vertices. For this result and more, refer to the standard Ref. Athreya and Ney (1972).

At time (or generation) 0, the GW tree $\tau(p)$ has a single vertex, the so-called *root*. At generation 0, the root branches several offsprings according to the offspring distribution p. We use M_n to denote the maximal number of offsprings of all vertices at generation n, and $M_{[0,n]}$ the maximal number of offsprings of all vertices in the first $n+1$ generations, including generation 0. Then clearly $\mathbf{P}[M_0 = r] = p_r$. We also use $M = M_{[0,\infty)}$ to denote the supremum of all outdegrees of $\tau(p)$. Clearly in the (sub)critical case M is finite a.s. and it is actually the maximum of all outdegrees. Use $F(r)$ to denote the distribution function of p, $\bar{F}(r)$ the tail function. Similarly we use $H(r)$ to denote the distribution function of M, $\bar{H}(r)$ the tail function. Note that in the supercritical case, it is possible for M to be infinite.

We use $G(x)$ to denote the generating function of p, that is, for $x \in [0, 1]$,

$$G(x) = p_0 + p_1 x + \cdots + p_n x^n + \cdots.$$

Clearly G is infinitely differentiable, and

$$\left.\frac{\partial G(x)}{\partial x}\right|_{x=1-} = \mu.$$

We write $F \circ G(x)$ for $F(G(x))$, $G^{\circ 2}(x)$ for $G \circ G(x)$, and similarly define $G^{\circ n}(x)$. By induction,

$$\left.\frac{\partial G^{\circ n}(x)}{\partial x}\right|_{x=1-} = \mu^n.$$

Finally for any nonnegative integer r, we use $G_r(x)$ to denote the generating function of p truncated at r, that is, for $x \in [0, 1]$,

$$G_r(x) = p_0 + p_1 x + \cdots + p_r x^r.$$

Next we give an exact expression of the finite-dimensional distribution of M_n at different generations. This result is very elementary, however here we give a general version with our applications later in mind.

Lemma 2.1. *Recall that M_i is the maximal outdegree at generation i. Then*

$$\mathbf{P}[M_i \le r_i, 0 \le i \le n] = G_{r_0} \circ G_{r_1} \circ \cdots \circ G_{r_n}(1).$$

Proof. For $n = 0$, trivially

$$\mathbf{P}[M_0 \le r_0] = p_0 + p_1 + \cdots + p_{r_0} = G_{r_0}(1).$$

For $n = 1$, by considering the outdegree of the root we get

$$\mathbf{P}[M_0 \le r_0, M_1 \le r_1] = p_0 + p_1 \mathbf{P}[M_0 \le r_1] + \cdots + p_{r_0} \mathbf{P}[M_0 \le r_1]^{r_0} = G_{r_0} \circ G_{r_1}(1).$$

The general case follows by induction and a similar identity

$$\mathbf{P}[M_i \le r_i, 0 \le i \le n] = p_0 + p_1 \mathbf{P}[M_{i-1} \le r_i, 1 \le i \le n] + \cdots + p_{r_0} \mathbf{P}[M_{i-1} \le r_i, 1 \le i \le n]^{r_0}. \quad \square$$

This lemma implies that

$$\mathbf{P}[M_n \le r] = G^{\circ n} \circ G_r(1) = G^{\circ n}(p_0 + \cdots + p_r).$$

In general, the generating function G has no explicit expression. However, we can get a simple asymptotic result of $\mathbf{P}[M_n = r]$ as $r \to \infty$.

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