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Nonlinear Lyapunov criteria for stochastic explosive solutions

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ABSTRACT

This paper deals with the problem of explosive solutions for a class of stochastic differential equations via Lyapunov's method. We give some sufficient conditions on the existence of explosive solutions for stochastic differential equations driven by Brownian motions.

1. Introduction

Consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sum_{r=1}^{n} \sigma_r(t, X(t))\mathrm{d}W_r(t), \\ X(t_0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $W_1(t), \dots, W_n(t)$ are independent standard Wiener processes, $b,\ \sigma_r : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d,\ r = 1, \dots, n$, are locally bounded and locally Lipschitz continuous.

In this paper, we are concerned with the explosion of solutions for problem (1.1). Feller (1954) first gave a sufficient condition for almost surely explosive solutions of one dimensional ($d = 1$) stochastic differential equations. For general case $d > 1$, Khasminskii (2012, 1960) discussed the explosive problem by Lyapunov's method. Some explosive problems of stochastic differential equations were quoted in the books by McKean (1969) and Ikeda and Watanabe (1989). Recently, Chow and Khasminskii (2014) found that the explosion or non-explosion of solutions depends mostly on the properties of coefficients for $|x| > R$ with any $R > 0$, provided $|x_0|$ is large enough. In the previous results, the Lyapunov function $V(t, x)$ needs to be bounded and satisfy the differential inequality $LV \geq CV$ for some positive constant C, where the operator L denotes the infinitesimal generator. In this paper we give nonlinear Lyapunov criteria by substituting a function $F(t, V)$ for CV, where $F(t, V)$ can be nonlinear in V.

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The main results are presented in Section 2. We first prove Theorem 2.1 in which solutions of (1.1) explode with positive probability if there exists a Lyapunov function $V(t,x)$ satisfying $LV \geq F(t,V)$, where $F(t,V) : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ may not be linear, for example $F(t,V) = CV^2$. On the basis of Bihari's inequality, we give Theorem 2.2. Then, we give two examples to illustrate the effectiveness of the two theorems. All these results provide more ways in studying explosive solutions of stochastic differential equations.

2. Main results

Let $\mathcal{C}^{1,2}([t_0,\infty)\times\mathbb{R}^d, J)$ denote the class of real-valued functions on $[t_0,\infty)\times\mathbb{R}^d$ which have continuous first derivatives in t and second derivatives in x. Then for $V(t,x) \in \mathcal{C}^{1,2}([t_0,\infty)\times\mathbb{R}^d, J)$, the infinitesimal generator L for $(t, X(t))$ has the form

$$LV(t,x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 V}{\partial x_i \partial x_j},$$

where $a_{ij} = \sum_{r=1}^{n} \sigma_{ri}\sigma_{rj}$ with $\sigma_r = (\sigma_{r1},\dots\sigma_{rd})^T$.

We denote by $|\cdot|$ the Euclidean norm, that is $|x| = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$, where $x = (x_1,\dots,x_d)$.

For a stochastic process $X(t)$, let $\tau_R = \inf\{t \geq t_0 : |X(t)| \geq R\}$, and $\tau_\infty = \lim_{R\to\infty}\tau_R$, $\tau_R(t) = \min\{\tau_R, t\}$.

The solution $X(t)$ of problem (1.1) is said to explode with positive probability, if $\mathbb{P}\{\tau_\infty < \infty\} > 0$.

Definition 2.1. Let U be an open (t,u)-set in $\mathbb{R}^2$, and $F \in \mathcal{C}(U,\mathbb{R})$. Consider the differential equation with an initial condition

$$\begin{cases} \mathrm{d}u = F(t,u)\mathrm{d}t, \\ u(t_0) = u_0. \end{cases} \tag{2.2}$$

The solution $r(t)$ is said to be a minimal (maximal) solution on $[t_0, t_0+a)$, if for every solution $u(t)$ on $[t_0, t_0+a)$, $u(t) \geq r(t)$ $(u(t) \leq r(t))$.

Lemma 2.1 (*See Lakshmikantham and Leela, 1969*)**.** *Let U be an open (t,u)-set in $\mathbb{R}^2$, and $F \in \mathcal{C}(U,\mathbb{R})$. Suppose that $F(t,u)$ is monotonic nondecreasing in u for each t. Let $m \in \mathcal{C}([t_0,t_0+a),\mathbb{R})$, $(t,m(t)) \in U$, $t \in [t_0,t_0+a)$, $m(t_0) \geq u_0$, and*

$$m(t) \geq m(t_0) + \int_{t_0}^{t} F(s,m(s))\mathrm{d}s, \quad t \in [t_0, t_0+a).$$

Then

$$m(t) \geq r(t), \quad t \in [t_0, t_0+a),$$

where $r(t)$ is the minimal solution of (2.2).

Theorem 2.1. *Suppose that the coefficients of (1.1) are continuous and satisfy the Lipschitz condition on any compact subset of $\mathbb{R}^d$ for any $t \geq t_0$. Moreover, there exists a bounded function $V(t,x) \in \mathcal{C}^{1,2}([t_0,\infty)\times\mathbb{R}^d, J)$ and a function $F \in \mathcal{C}([t_0,\infty)\times J,\mathbb{R})$ which is convex and non-decreasing in the second variable, such that*

$$LV \geq F(t,V),$$

and the minimal solution of the following differential equations with initial condition

$$\begin{cases} \mathrm{d}u = F(t,u)\mathrm{d}t, \\ u(t_0) = V(t_0,x_0), \end{cases} \tag{2.3}$$

tends to ∞, when $t \to T$ ($T \geq t_0$ can be a finite value or infinity). Then the solution $X(t)$ of (1.1) explodes with positive probability, more precisely, we have $\mathbb{P}(\tau_\infty < T) > 0$.

Proof. It is obvious that the solutions of problem (1, 1) exist. By Itô's formula, we have

$$\mathbf{E}\Big(V\big(\tau_n(t), X(\tau_n(t))\big) - V(t_0,x_0)\Big) = \mathbf{E}\Big(\int_{t_0}^{\tau_n(t)} LV\big(s, X(s)\big)\mathrm{d}s\Big).$$

Let $n \to \infty$ and $\tau(t) = \min\{\tau_\infty, t\}$. We have

$$\begin{aligned}\mathbf{E}\Big(V\big(\tau(t), X(\tau(t))\big) - V(t_0,x_0)\Big) &= \mathbf{E}\Big(\int_{t_0}^{\tau(t)} LV\big(s, X(s)\big)\mathrm{d}s\Big) \\ &\geq \mathbf{E}\left(\int_{t_0}^{\tau(t)} F(s, V(s))\mathrm{d}s\right).\end{aligned}$$