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Weak convergence of equity derivatives pricing with default risk[☆]

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ABSTRACT

This paper presents a discrete-time equity derivatives pricing model with default risk in a no-arbitrage framework. Using the equity-credit reduced form approach where default intensity mainly depends on the firm's equity value, we deduce the Arrow–Debreu state prices and the explicit pricing result in discrete time after embedding default risk in the pricing model. We prove that the discrete-time defaultable equity derivatives pricing has convergence stability, and it converges weakly to the continuous-time pricing results.

1. Introduction

Default risk is the risk that the agents cannot fulfill their obligations in the contracts. The reduced form approach has become a standard tool for modeling default risk. It considers the default to be an exogenously specified jump process, and derives the default probability as the instantaneous likelihood of default, see, for example, Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Lando (1998). The default time is usually defined as the first jump time of a Cox process with a given intensity (hazard rate). Hence, these models are frequently called *intensity models*.

Recently, an alternative model named *equity-credit market approach* has emerged. It assumes that the default intensity depends on the firm's equity value (stock prices) and allows the stock price to jump to zero at the time of default. It has both reduced form and structural features. Default risk is incorporated in this equity modeling approach by assuming that the stock price S_t at time t can jump to zero with an intensity, which is assumed to be a function of S_t. The models described above are all continuous-time models, they are widely used to model default risk.

However, continuous-time models are often too complicated to handle, it is necessary to deduce discrete-time models and show that the pricing processes converge to the continuous-time models. This is not a trivial job, since weak convergence, by its nature, is not tied to a single probability space. Some authors have presented different discrete-time models for derivatives pricing and have established some weak convergence results. See, for example, Cox and Rubinstein (1979), He (1989), Duffie and Protter (1992) and Nieuwenhuis and Vellekoop (2004), etc.

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In this paper, our aim is to present a discrete-time equity derivatives pricing model with default risk in a no-arbitrage framework, and prove that the pricing in discrete-time converges weakly to the continuous-time pricing results. In comparison, our method is different from Nieuwenhuis and Vellekoop (2004). Following the discrete framework of He (1989) and equity-credit market approach presented in Bielecki et al. (2009), we describe the discrete-time pricing model in a no-arbitrage framework. After embedding default risk, we deduce the Arrow–Debreu state prices and the explicit pricing result in discrete time. In order to prove the weak convergence of pricing processes, several auxiliary results are presented.

The paper is organized as follows: in Section 2, we introduce the continuous-time model using equity-credit reduced-form approach; In Section 3, we illustrate a discrete-time model of the equity derivatives pricing with default risk; In Section 4, weak convergence of equity derivatives pricing with default risk from discrete-time to continuous-time pricing is proved; Finally, in Section 5, we summarize the article and make concluding remarks.

2. The continuous-time model

We first recall the continuous-time defaultable contingent claims pricing model. Given a probability space $(\Omega, \mathcal{F}, P)$, T is a strictly positive real number which represents the final date, $(\omega_t)_{0\le t\le T}$ is a Brownian motion. Let $\mathcal{F}_t = \sigma(\omega_s, s \le t)$ for $t \ge 0$. We suppose $\mathcal{F}_t \subset \mathcal{F}$ for all t, and P is the real-world probability. Furthermore, we denote by "$\Rightarrow$" weak convergence from now on.

A default event occurs at a random time τ, where τ is a non-negative random variable. The default process is defined as $N_t \triangleq \mathbf{1}_{\{\tau\le t\}}$, and $\mathcal{H}_t = \sigma(N_s, s \le t)$, the filtration $\mathcal{H}$ is used to describe the information about default time, where $\mathcal{H} = \bigcup_{0\le t\le T} \mathcal{H}_t$. At any time t, the agent's information on the securities prices and default time is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathcal{G}$ stopping time where $\mathcal{G} = \bigcup_{0\le t\le T} \mathcal{G}_t$. In fact, $\mathcal{G}$ is the smallest filtration which contains $\mathcal{F}$ and allows τ to be a stopping time. Assume that the pre-default stock price S_t has the following dynamics

$$dS_t = (b(S_t) + \lambda(S_t, t)S_t)dt + \sigma(S_t, t)S_t d\omega_t, \quad S_0 > 0. \tag{2.1}$$

Here we assume that $b(x)$ is continuous, $\sigma(S, t)$ is a positively bounded and nonsingular Borel-measurable function. In particular we have that $\sigma(S, t) \ge \sigma$ for some positive constant σ, $\lambda(S, t)$ is a nonnegative, bounded, continuous, $\mathcal{F}-$progressively measurable and integrable function. The functions $b(S)$, $\lambda(S, t)S$ and $\sigma(S, t)S$ are Lipschitz continuous in S, uniformly in t.

The bond price B_t satisfies $dB_t = B_t r(S_t)dt$ and $B_0 = 1$, where $r(x)$ is a nonnegative continuous function, representing the riskless interest rate. Suppose there exists a constant $K > 0$ such that $|x^2 r(x)| \le K(1 + x^2)$.

There exists a $\mathcal{G}$ equivalent martingale measure Q^* which is defined as $dQ^*|_{\mathcal{F}_t} = \xi_t dP|_{\mathcal{F}_t}$, where ξ_t is the Radon–Nikodým density satisfying

$$d\xi_t = \xi_t \theta(S_t)d\omega_t, \quad \xi_0 = 1. \tag{2.2}$$

Here $\theta(x) = -\sigma(x)^{-1}(b(x) - r(x)x)$. Define W_t via $dW_t = d\omega_t - \theta(S_t)dt$, then W_t is a Brownian motion with respect to $\mathcal{F}$, and under the changed measure

$$dS_t = S_t[(r(S_t) + \lambda(S_t, t))dt + \sigma(S_t, t)dW_t], \quad S_0 > 0. \tag{2.3}$$

Define $G_t \triangleq Q^*(\tau > t \mid \mathcal{F}_t)$, $\Gamma_t \triangleq -\ln G_t$. We call Γ_t the $\mathcal{F}$ *hazard process of* τ. For the detailed properties, one can refer to Bielecki and Rutkowski (2002).

Let $g(\cdot) : \mathbb{R} \to \mathbb{R}$ be a square integrable and measurable function, the equity derivatives are defined to be securities that pay $g(S_T)$ dollars on the final date. This formulation subsumes all of the usual examples, such as the European options, convertible bonds and so on. The prices of equity derivatives at time t are

$$V(S_t, t) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{Q^*}\left[\frac{B_t e^{\Gamma_t}}{B_T e^{\Gamma_T}} g(S_T) \,\middle|\, \mathcal{F}_t \right]. \tag{2.4}$$

Poisson process with stochastic intensity is called *Cox process*. Given $\lambda(S_u, u)$, denote by $\{\overline{C}_t\}$ the Poisson process with intensity $C_t = \int_0^t \lambda(S_u, u)du$. Then $\{\overline{C}_t\}$ is a Cox process. Following the equity-credit market models, the canonical construction of default time τ under the Cox process $\{\overline{C}_t\}$ is defined as $\tau = \inf\{t \ge 0 : C_t \ge \Theta\}$, where $\Theta \sim Exp(1)$ and is independent of $\mathcal{F}$ under Q^*. Then

$$Q^*(\tau > t \mid \mathcal{F}_t) = Q^*(\Theta > C_t \mid \mathcal{F}_t) = e^{-C_t}.$$

It is easy to see that under this condition, the default time is the first jump time of the Cox process, the $\mathcal{F}$ hazard process of τ satisfies

$$\Gamma_t = -\ln Q^*(\tau > t \mid \mathcal{F}_t) = -\ln Q^*(\Theta > C_t \mid \mathcal{F}_t) = C_t.$$

Let Δ denote the bankruptcy state when the firm defaults at time τ. Then we can also write the dynamics for the stock price subject to bankruptcy S_t^Δ as follows:

$$dS_t^\Delta = S_t^\Delta[r(S_t)dt + \sigma(S_t, t)dW_t - dM_t],$$