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On Bayesian asymptotics in stochastic differential equations with random effects

Trisha Maitra, Sourabh Bhattacharya *

Interdisciplinary Statistical Research Unit, Indian Statistical Institute, 203, B. T. Road, Kolkata 700108, India

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ABSTRACT

Delattre et al. (2013) investigated asymptotic properties of the maximum likelihood estimator of the population parameters of the random effects associated with n independent stochastic differential equations (*SDE*'s) assuming that the *SDE*'s are independent and identical (*iid*).

In this article, we consider the Bayesian approach to learning about the population parameters, and prove consistency and asymptotic normality of the corresponding posterior distribution in the *iid* set-up as well as when the *SDE*'s are independent but non-identical.

1. Introduction

Mixed effects models are appropriate when dealing with data sets consisting of variability between subjects and also within subjects, with respect to time. Although a great deal of work on mixed effects models exists in the statistical literature, mixed effects models where within subject variability is modeled via stochastic differential equations (*SDE*'s) are relatively rare. For a relatively short but comprehensive review we refer the reader to Delattre et al. (2013), who also undertake theoretical and asymptotic investigation of a class of *SDE*-based mixed effects models having the following form: for $i = 1, \ldots, n$,

$$dX_i(t) = b(X_i(t), \phi_i)dt + \sigma(X_i(t))dW_i(t), \tag{1.1}$$

where, for $i = 1, \ldots, n$, $X_i(0) = x^i$ is the initial value of the stochastic process $X_i(t)$, which is assumed to be continuously observed on the time interval $[0, T_i]$; $T_i > 0$ assumed to be known. The function $b(x, \varphi)$ is a known, real-valued function on $\mathbb{R} \times \mathbb{R}^d$ ($\mathbb{R}$ is the real line and d is the dimension); this function is known as the drift function. The function $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is the known diffusion coefficient. In the context of statistical modeling, $X_i(\cdot)$ models the ith individual. The *SDE*'s given by (1.1) are driven by independent standard Wiener processes $\{W_i(\cdot);\ i = 1, \ldots, n\}$, and $\{\phi_i;\ i = 1, \ldots, n\}$, which are to be interpreted as the random effect parameters associated with the n individuals, are assumed to be independent of the Brownian motions and independently and identically distributed (*iid*) random variables with common distribution $g(\varphi, \theta)d\nu(\varphi)$. Here $g(\varphi, \theta)$ is a density with respect to a dominating measure on $\mathbb{R}^d$, for all θ, where $\theta \in \Omega \subset \mathbb{R}^p$ ($p \geq 2d$) is the unknown parameter of interest, which is to be estimated. Delattre et al. (2013) impose regularity conditions that ensure existence of solutions of (1.1). The conditions, which are also adopted by us, are as follows.

* Corresponding author.
E-mail address: sourabh@isical.ac.in (S. Bhattacharya).

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(H1) (i) The function $(x,\varphi)\mapsto b(x,\varphi)$ is C^1 (differentiable with continuous first derivative) on $\mathbb{R}\times\mathbb{R}^d$, and such that there exists $K>0$ so that
$$b^2(x,\varphi)\le K(1+x^2+|\varphi|^2),$$
for all $(x,\varphi)\in\mathbb{R}\times\mathbb{R}^d$.

(ii) The function $\sigma(\cdot)$ is C^1 on $\mathbb{R}$ and
$$\sigma^2(x)\le K(1+x^2),$$
for all $x\in\mathbb{R}$.

(H2) Let X_i^{φ} be associated with the SDE of the form (1.1) with drift function $b(x,\varphi)$. Also letting $Q_{\varphi}^{x^i,T_i}$ denote the joint distribution of $\left\{X_i^{\varphi}(t);\ t\in[0,T_i]\right\}$, it is assumed that for $i=1,\ldots,n$, and for all φ,φ', the following holds:
$$Q_{\varphi}^{x^i,T_i}\left(\int_0^{T_i}\frac{b^2\left(X_i^{\varphi}(t),\varphi'\right)}{\sigma^2(X_i^{\varphi}(t))}dt<\infty\right)=1.$$

(H3) For $f=\frac{\partial b}{\partial\varphi_j}$, $j=1,\ldots,d$, there exist $c>0$ and some $\gamma\ge 0$ such that
$$\sup_{\varphi\in\mathbb{R}^d}\frac{|f(x,\varphi)|}{\sigma^2(x)}\le c\left(1+|x|^{\gamma}\right).$$

In this article, we consider $d=1$, that is, we assume one-dimensional random effects, so that $\varphi\in\mathbb{R}$. Moreover, as in Delattre et al. (2013), for statistical inference we assume that $b(x,\phi_i)$ is linear in ϕ_i; in other words, $b(x,\phi_i)=\phi_i b(x)$. Under this assumption, (H3) is not required; see Delattre et al. (2013) and Maitra and Bhattacharya (2014a). Following Maitra and Bhattacharya (2014a) we further assume that

(H1′) $b(\cdot)$ and $\sigma(x)$ are C^1 on $\mathbb{R}$ satisfying $b^2(x)\le K(1+x^2)$ and $\sigma^2(x)\le K(1+x^2)$ for all $x\in\mathbb{R}$, for some $K>0$.

(H2′) Almost surely for each $i\ge 1$,
$$\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}ds<\infty.$$

As in Delattre et al. (2013) and Maitra and Bhattacharya (2014a) here we assume that ϕ_i are normally distributed implying for $k\ge 1$, $E|\phi_i|^{2k}<\infty$ so that the following holds for all $T>0$ (see Delattre et al., 2013):
$$\sup_{t\in[0,T]}E\left[X_i(t)\right]^{2k}<\infty. \tag{1.2}$$

In fact, the linearity assumption $b(x,\phi_i)=\phi_i b(x)$ and the assumption that ϕ_i are Gaussian random variables are crucial for availability of an explicit form of the likelihood of the parameters of the random effects ϕ_i. Indeed, assuming that $g(\varphi,\theta)d\nu(\varphi)\equiv N\left(\mu,\omega^2\right)$, Delattre et al. (2013) obtain the likelihood as the product of the following:
$$f_i(X_i|\theta)=\frac{1}{\left(1+\omega^2V_i\right)^{1/2}}\exp\left[-\frac{V_i}{2\left(1+\omega^2V_i\right)}\left(\mu-\frac{U_i}{V_i}\right)^2\right]\exp\left(\frac{U_i^2}{2V_i}\right), \tag{1.3}$$
where $\theta=(\mu,\omega^2)\in\mathbb{R}\times\mathbb{R}^+$ ($R^+=(0,\infty)$), and
$$U_i=\int_0^{T_i}\frac{b(X_i(s))}{\sigma^2(X_i(s))}dX_i(s),\qquad V_i=\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}ds;\quad i=1,\ldots,n, \tag{1.4}$$
are sufficient statistics. In (1.3), for $i=1,\ldots,n$, X_i stands for $\{X_i(t);\ t\in[0,T_i]\}$.

Delattre et al. (2013) consider the *iid* set-up by setting $x^i=x$ and $T_i=T$ for $i=1,\ldots,n$, and directly prove weak consistency (convergence in probability) and asymptotic normality of the *MLE* of θ. As an alternative, Maitra and Bhattacharya (2014a) verify the regularity conditions of existing results in general set-ups provided in Schervish (1995) and Hoadley (1971) to prove asymptotic properties of the *MLE* in this *SDE* set-up. In the *iid* set-up, this approach allowed Maitra and Bhattacharya (2014a) to establish strong consistency of the *MLE*, rather than weak consistency. Moreover, assumption (H4) of Delattre et al. (2013), requiring that $b(\cdot)/\sigma(\cdot)$ is non-constant and for $i\ge 1$, (U_i,V_i) admits a density with respect to the Lebesgue measure on $\mathbb{R}\times\mathbb{R}^+$ which is jointly continuous and positive on an open ball of $\mathbb{R}\times\mathbb{R}^+$, was not required in their approach. Also, not only in the *iid* situation, Maitra and Bhattacharya (2014a) prove asymptotic results related to the *MLE* even in the independent but non-identical (we refer to this as non-*iid*) case.

To our knowledge, Bayesian asymptotics has not been investigated in the context of mixed effects models, even though applied Bayesian analysis of such models is not rare (see, for example, Wakefield et al., 1994; Wakefield, 1996; Bennett et al., 1996). In this article, we consider the Bayesian framework associated with *SDE*-based random effects model, for both *iid* and