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Smooth densities of the laws of perturbed diffusion processes

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ABSTRACT

Under some regularity conditions on b, σ and α, we prove that the solution of the following perturbed stochastic differential equation

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \alpha \sup_{0\le s\le t} X_s, \quad \alpha < 1 \tag{0.1}$$

admits smooth densities for all $0 < t \le t_0$, where $t_0 > 0$ is some finite number.

1. Introduction

There have been a considerable body of literatures devoted to the study of perturbed stochastic differential equations (SDEs), see Carmona et al. (1998), Chaumont and Doney (1999), Davis (1999), Doney (1998), Doney and Zhang (2005), Le Gall and Yor (1986, 1990), Perman and Werner (1997), Werner (1995) and Yue and Zhang (2015). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a filtered probability space with filtration$\{\mathcal{F}_t\}_{t\ge0}$, let $\{B_t\}_{t\ge0}$ be a one-dimensional standard $\{\mathcal{F}_t\}_{t\ge0}$-Brownian Motion. Suppose that $\sigma(x)$, $b(x)$ are Lipschitz continuous functions on $\mathbb{R}$. It was proved in Doney and Zhang (2005) that the following perturbed stochastic differential equation:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \alpha \sup_{0\le s\le t} X_s, \quad \forall \alpha < 1, \tag{1.1}$$

admits a unique solution. If $|\sigma(x)| > 0$, it was shown in Yue and Zhang (2015) that the law of X_t is absolutely continuous with respect to Lebesgue measure, i.e. the law of X_t admits a density for $t > 0$.

There seem no results on the smoothness of the density of the law of a perturbed diffusion process. This paper aims to partly fill in this gap. The smoothness of densities is a popular topic in stochastic analysis and has been intensively studied for several decades. We refer readers to Nualart (2006), Sanz-Sole (2005) and references therein. Our approach to proving the smoothness of densities is by Malliavin calculus, so let us first recall some well known results on Malliavin calculus (Nualart, 2006) to be used in this paper.

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Let $\Omega = C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ which are zero at zero. Denote by $\mathcal{F}$ the Borel σ-field on Ω and $\mathbb{P}$ the Wiener measure, then the canonical coordinate process $\{\omega_t, t \in \mathbb{R}_+\}$ on Ω is a Brownian motion B_t. Define $\mathcal{F}_t^0 = \sigma(B_s, s \le t)$ and denote by $\mathcal{F}_t$ the completion of $\mathcal{F}_t^0$ with respect to the $\mathbb{P}$-null sets of $\mathcal{F}$.

Let $H := L^2(\mathbb{R}_+, \mathcal{B}, \mu)$ where $(\mathbb{R}_+, \mathcal{B})$ is a measurable space with $\mathcal{B}$ being the Borel σ-field of $\mathbb{R}_+$ and μ being the Lebesgue measure on $\mathbb{R}_+$. We denote the norm of H by $\|\cdot\|_H$. For any $h \in H$, $W(h)$ is defined by

$$W(h) = \int_0^\infty h(t)\mathrm{d}B_t. \tag{1.2}$$

Note that $\{W(h), h \in H\}$ is a Gaussian Process on H.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let $\mathcal{S}$ be the set of smooth random variables defined by

$$\mathcal{S} = \{F = f(W(h_1), \dots, W(h_n));\ h_1, \dots, h_n \in H, n \ge 1, f \in C_p^\infty(\mathbb{R}^n)\}.$$

Let $F \in \mathcal{S}$, define its Malliavin derivative D_tF by

$$D_tF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n))h_i(t), \tag{1.3}$$

and its norm by

$$\|F\|_{1,2} = [\mathbb{E}(|F|^2) + \mathbb{E}(\|DF\|_H^2)]^{\frac{1}{2}},$$

where $\|DF\|_H^2 = \int_0^\infty |D_tF|^2\mu(\mathrm{d}t)$. Denote by $\mathbb{D}^{1,2}$ the completion of $\mathcal{S}$ under the norm $\|\cdot\|_{1,2}$. We further define the norm

$$\|F\|_{m,2} = \left[\mathbb{E}(|F|^2) + \sum_{k=1}^m \mathbb{E}(\|D^kF\|_{H^{\otimes k}}^2)\right]^{\frac{1}{2}}.$$

Similarly, $\mathbb{D}^{m,2}$ denotes the completion of $\mathcal{S}$ under the norm $\|.\|_{m,2}$.

We shall use the following two propositions:

Proposition 1.1 (*Proposition 1.2.3 of Nualart, 2006*). *Let $\phi : \mathbb{R}^d \to R$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \dots, F^d)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$, and*

$$D(\phi(F)) = \sum_{i=1}^d \partial_i\phi(F)DF^i.$$

Proposition 1.2 (*Proposition 2.1.5 of Nualart, 2006*). *If $F \in \mathbb{D}^{\infty,2}$ with $\mathbb{D}^{\infty,2} = \cap_{m\ge1}\mathbb{D}^{m,2}$ and $\|DF\|_H^{-1} \in \cap_{p\ge1}L^p(\Omega)$, then the density of F belongs to the space $C^\infty(\mathbb{R})$ of infinitely continuously differentiable functions.*

Throughout this paper, for a bounded measurable function f, we shall denote

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

2. Main results

Throughout this paper, we need to assume $\alpha < 1$ to guarantee that Eq. (1.1) has a unique solution Doney and Zhang (2005). Furthermore, it is shown in Yue and Zhang (2015) that

Theorem 2.1 (*Yue and Zhang, 2015, Theorem 3.1*). *Let $(X_t)_{t\ge0}$ be the unique solution to Eq. (1.1). Then $X_t \in \mathbb{D}^{1,2}$ for all $t > 0$.*

Theorem 2.2 (*Yue and Zhang, 2015, Theorem 3.2*). *Assume that σ and b are both Lipschitz continuous, and $|\sigma(x)| > 0$ for all $x \in \mathbb{R}$. Then, for $t > 0$, the law of X_t is absolutely continuous with respect to Lebesgue measure.*

In this paper, we shall prove the following results about the smoothness of densities:

Theorem 2.3. *Assume that b is bounded smooth with $\|b'\|_\infty < \infty$ and that $\sigma(x) \equiv \sigma$. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, b) < 1/2,$$

with $\theta(t_0, \alpha, b) := \sqrt{2\|b'\|_\infty^2 t_0^2 + 8\alpha^2} + \|b'\|_\infty^2 t_0^2 + 4\alpha^2$, then the law of X_t in (1.1) admits a smooth density for all $t \in (0, t_0]$.