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Sharp bounds on DMRL and IMRL classes of life distributions with specified mean

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ABSTRACT

We obtain sharp upper and lower bounds for a reliability function with decreasing mean residual life (DMRL), in terms of its mean. The constructive proofs establish that the bounds are sharp. We also provide bounds for the IMRL class.

1. Introduction

Suppose F is a distribution function defined over the interval $[0, \infty)$ and $\bar{F} = 1-F$ is the corresponding reliability/survival function. The distribution F is said to be "decreasing mean residual life" (DMRL) if its mean residual life function, defined as

$$m(t) = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)}, \quad t \geq 0 \tag{1}$$

is a non-increasing function. The DMRL property of a distribution is an intuitively meaningful notion of ageing. Therefore, the class of DMRL life distributions, introduced by Bryson and Siddiqui (1969), has been considered by reliability theorists, demographers, actuarial scientists and biometrists as an important class of ageing distributions. The DMRL class of life distributions contains the "increasing failure rate" (IFR) class of distributions (Bryson and Siddiqui, 1969) and is contained in the "new better than used in expectation" (NBUE) class of distributions (Deshpande et al., 1986). It has partial overlap with the "increasing failure rate average" (IFRA) (Bryson and Siddiqui, 1969) and the "new better than used in" (NBU) classes of distributions. The DMRL class is closed under formation of parallel systems of independent and identically distributed components (Abouammoh and El-Neweihi, 1986), but not closed under convolution, formation of general coherent systems or mixtures (Park, 2003). Klefsjö (1982b) has provided a characterization of the DMRL class through the total time on test (TTT) transform. Various useful properties of the DMRL class of life distributions may be found in Balaban and Singpurwalla (1984) and Willmot and Lin (2001); Korczak (2001). Definitions, properties and interrelations of the different ageing classes may be found in Barlow and Proschan (1981), Gertsbakh (1989), Lai and Xie (2006) and Gupta et al. (2010). Many researchers have proposed tests of exponentiality of a distribution against the DMRL alternative (see Hollander and Frank, 1975, Klefsjö, 1983, Bergman and Klefsjö, 1989, Ahmad, 1992, Lim and Park, 1997, Abu-Youssef, 2002, Ahmad and Mugdadi, 2004, Anis, 2010). On the other hand the dual of the DMRL class, i.e., the "increasing mean residual life" (IMRL) class, contains the

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"decreasing failure rate" (DFR) class of distributions and is contained in the "new worse than used in expectation" (NWUE) class of distributions. It has partial overlap with the "decreasing failure rate average" (DFRA) and the "new worse than used" (NWU) classes of distributions and is closed under formation of mixtures of life distributions (Park, 2003).

The main motivation for consideration of a nonparametric class of distributions with an ageing property (such as IFR or DMRL) lies in the common properties enjoyed by members of that class. In particular, bounds on the reliability of an ageing unit, whose life distribution belongs to such a class, have been a topic of great interest among reliability theorists. The interest lies in finding the sharpest possible upper and lower bounds on $\bar{F}(t)$ for a given t, assuming that the lifetime distribution belongs to a class such as IFR or NBU with a specified mean (or a specified quantile or higher moment). Various results in this area are available in the literature. Upper and lower bounds of IFR and IFRA life distributions with a specified mean are available in the classic textbook by Barlow and Proschan (1981). Lower bounds on the reliability function for the NBU and the NBUE classes of life distributions can be found in Marshall and Proschan (1972). Korzeniowski and Opawski (1976) derived an upper bound on the reliability function for the NBU class. Haines et al. (1974) provided an upper bound of the NWUE class of reliability functions. Klefsjö (1982a) has provided both the upper and lower bounds of the reliability functions of the "harmonically new better than used in expectation" (HNBUE) class and its dual, the "harmonically new worse than used in expectation" (HNWUE) class. These results are proved using different techniques and the sharpness of these bounds have been established. Sengupta (1994) derived the sharpest upper and lower bounds on the reliability of IFR, IFRA, NBU, DFR, DFRA and NWU classes of life distributions, in a unified way. Bounds based on other moments and quantiles are also available. One can see Lai and Xie (2006) for a compilation of results on this topic.

Unfortunately, there is no analogous result for the DMRL and IMRL classes. Cheng and He (1989a) has provided a bound on the gap between a DMRL reliability function and an exponential reliability function with equal mean. This inequality may be used to obtain bounds on a DMRL reliability function. The lower bound on NBUE reliability function given in Marshall and Proschan (1972) and the upper bound given in Klefsjö (1982a) for HNBUE reliability functions are applicable to DMRL reliability functions. However, these bounds may not be sharp.

In this paper, the sharp upper and lower bounds of a reliability function $\bar{F}$ belonging to the DMRL class are obtained in terms of its mean $\mu = m(0) = \int_0^\infty \bar{F}(t)dt$. This is achieved by following the strategy of Sengupta (1994), i.e., by identifying a tractable and parametric subclass over which the optimization would suffice. Results for the negatively ageing class of IMRL distributions are also derived in the same manner.

2. Lower bound on DMRL reliability function

The reliability function $\bar{F}(t)$ can be expressed in terms of the corresponding mean residual life (MRL) function of a life distribution through the relation

$$\bar{F}(t) = \frac{m(0)}{m(t)} e^{-\int_0^t \frac{du}{m(u)}}.$$

This representation and the following property of the MRL function would be useful in the sequel.

Lemma 1. *If $m(t)$ is the MRL of a life distribution, then $m'(t) \geq -1$.*

Proof. By differentiating (1), we get

$$\begin{aligned} m'(t) &= \frac{-\bar{F}^2(t) + f(t)\int_t^\infty \bar{F}(u)du}{\bar{F}^2(t)} \\ &= -1 + r(t)m(t), \end{aligned}$$

where $r(t)$ is the hazard rate of the distribution F. The stated result follows. □

Let $\mathcal{D}$ be the set of all DMRL reliability functions with mean μ, that is,

$$\mathcal{D} = \left\{ \bar{F} : m(t) \downarrow t, \int_0^\infty \bar{F}(u)du = \mu \right\}. \tag{2}$$

We consider the family of reliability functions $\bar{F}_a(t)$ defined as

$$\bar{F}_a(t) = \begin{cases} 1 & \text{for } 0 \leq t < a, \\ e^{-\frac{t-a}{\mu-a}} & \text{for } a \leq t, \end{cases} \tag{3}$$

for $a \geq 0$. It may be easily verified that the corresponding mean residual life function is

$$m_a(t) = \begin{cases} \mu - t & \text{for } 0 \leq t < a, \\ \mu - a & \text{for } a \leq t. \end{cases}$$

Clearly, $m_a(0) = \mu$. Therefore, $\bar{F}_a$ is a member of $\mathcal{D}$ for all $a \in [0, (t \wedge \mu)]$.

The next lemma shows that minimization of $\bar{F}(t)$ over $\mathcal{D}$ is equivalent to its minimization over the sub-class of reliability functions of the form $\bar{F}_a$.