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Cramér type moderate deviations for the number of renewals☆

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ABSTRACT

For the renewal process whose life-length variables are independent and identically distributed and satisfying the Cramér condition, we establish a moderate deviation estimation for it.

1. Introduction

Renewal process is an important kind of stochastic processes and it can be used to analyze many real problems such as predicting traffic flows, estimating the risk of earthquakes and so on. Having a better understanding of the process facilitates us to solve these problems.

We have known some asymptotic properties of the process. Billingsley (1968) has proven that renewal process is asymptotically normally distributed. Gunnar (1980) exhibited a remainder term estimate for the approximation. His result plays an important role in the proof of our result, we will illustrate it in detail later.

Moderate deviations date back to Cramér (1938) who obtained expansions for tail probabilities for sums of independent random variables. Petrov (1975) presented many asymptotic relations connected with Cramér series. In this paper, we establish a Carmér type moderate deviation for the renewal process.

2. Main result

Before stating the main result of the paper, we introduce the fundamental notations used in the paper. Let $\{X_i,\ i = 1, 2, \ldots\}$ be independent and identically distributed non-negative random variables with

$$\mu = EX_1, \qquad \sigma^2 = D^2X_1, \qquad \gamma^3 = E|X_1 - \mu|^3, \tag{1}$$

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where μ, σ and γ are finite constants. In the paper, we focus on renewal process $N(t)$ which is the number of renewals during the time interval $[0, t]$, i.e.

$$N(t) = \max\left\{n : \sum_{j=1}^{n} X_j \le t\right\}, \quad t \ge 0. \tag{2}$$

Φ denotes the standard normal distribution function.

Our main result is the following theorem.

Theorem 1. *Assume that X_1 satisfies the Cramér condition, namely, there exists $t_1 > 0$ such that $Ee^{t_1|X_1|} \le C < \infty$. Then for any fixed positive constant c_0 and for any fixed t*

$$\frac{P\left(\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \ge x\right)}{1-\Phi(x)} = 1 + O(1)\frac{1+x^3}{\sqrt{t}}, \tag{3}$$

holds uniformly for $0 < x \le c_0 t^{1/6}$.

To conclude with the section, we introduce some important results that are used to prove the theorem. The first theorem is provided by Gunnar (1980), and we state his result as below.

Theorem 2.1. *With notations as above we have for $t \ge 0$*

$$\sup_n \left| P(N(t) < n) - \Phi\left(\frac{(n\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}}\right)\right| \le 4\left(\frac{\gamma}{\sigma}\right)^3 \frac{\sqrt{\mu}}{\sqrt{t}}. \tag{4}$$

Note that if a random variable X satisfies the Cramér condition that the moment-generating function of X exists, then its moments of all orders exist. Thus by Taylor expansions and Petrov's (1975, Chapter 8, equation (2.41) and (2.42)) result, we can deduce the following theorem.

Theorem 2.2. *Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $EX_1 = 0$ and $Var(X_1) = 1$ such that $Ee^{t_0|X_1|} \le C < \infty$ for some $t_0 > 0$, then for $0 \le x \le a_0 n^{1/6}$, we have*

$$\frac{P(W_n \le -x)}{1-\Phi(x)} = 1 + O(1)\frac{1+x^3}{\sqrt{n}}, \tag{5}$$

where $W_n = (X_1 + \cdots + X_n)/\sqrt{n}$, a_0 is any fixed positive constant and $O(1)$ is bounded by a constant depending on C and t_0.

3. Proof of Theorem 1

First of all, fix a positive constant c_0, we suppose that $0 \le x \le c_0 t^{\frac{1}{6}}$. In the beginning, we reduce the problem by showing that Eq. (3) is almost trivial for $0 < t \le M = \max\{2, 2\mu, \frac{2\mu^3}{\sigma^2} + \mu\}$. In this case, for every fixed $c_0 > 0$, $0 \le x \le c_0 M^{\frac{1}{6}}$. The left part of (3) is bounded by a constant depending on c_0 and μ, since

$$\frac{P\left(\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \ge x\right)}{1-\Phi(x)} \le \frac{1}{1-\Phi(x)} \le \frac{1}{1-\Phi\left(c_0 M^{\frac{1}{6}}\right)}, \qquad \frac{1}{\sqrt{M}} \le \frac{1+x^3}{\sqrt{t}}.$$

Obviously, Eq. (3) holds. Now we divide the problem into two cases according to the range of x. In the first case, we suppose that $0 \le x \le \frac{\mu^{\frac{3}{2}}}{\sigma\sqrt{t}}$, and in the second case $\frac{\mu^{\frac{3}{2}}}{\sigma\sqrt{t}} \le x \le c_0 t^{\frac{1}{6}}$. To simplify the presentations, we let A be $\frac{\mu^{\frac{3}{2}}}{\sigma}$. The discussion is relatively easy in the first case and we can get the result directly.

We set $n(t, x)$ as the integer part of $x\sigma\sqrt{\frac{t}{\mu^3}} + \frac{t}{\mu}$ and let δ be the decimal part of $x\sigma\sqrt{\frac{t}{\mu^3}} + \frac{t}{\mu}$, namely

$$n(t, x) = \left[x\sigma\sqrt{\frac{t}{\mu^3}} + \frac{t}{\mu}\right] = x\sigma\sqrt{\frac{t}{\mu^3}} + \frac{t}{\mu} - \delta. \tag{6}$$

Notice that $N(t)$ can only be integer, we have

$$P\left(\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \ge x\right) = P\left(N(t) \ge x\sigma\sqrt{\frac{t}{\mu^3}} + \frac{t}{\mu}\right) = P\left(N(t) \ge n(t, x)\right). \tag{7}$$