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Convergence of moments for strictly stationary sequences

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ABSTRACT

For a class of strictly stationary sequences, we prove the convergence of moments.

1. Result

Let $\{X_k\}_{k\in\mathbb{Z}}, \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ be a strictly stationary random sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ and $S_n = \sum_{k=1}^{n} X_k$. We say that $\{X_k\}$ is in the domain of attraction of a law μ ($\{X_k\} \in \mathcal{DA}(\mu)$) if

$$\mathcal{L}(b_n^{-1}S_n - A_n) \to \mu \quad \text{as } n \to \infty, \tag{1.1}$$

for some strictly positive sequence $b_n \to \infty$ and a real sequence A_n. It follows from Theorem 3.1 in Jakubowski (1993) that if μ in (1.1) is non-degenerate and $A_n \equiv 0$ then there exists $\alpha \in (0, 2]$ such that $b_n^\alpha = nh(n)$, where $h(n)$ is a slowly varying sequence, and μ is strictly α-stable, if and only if the following condition holds:

$$\max_{1\le k+l\le n} |\mathrm{Cov}[\exp\{i\theta b_n^{-1}S_l\}, \exp\{i\theta b_n^{-1}S_k\}]| = o(1), \quad \theta \in \mathbb{R} \tag{1.2}$$

(see also Theorem 12.6 on pp. 397–398, vol. I in Bradley, 2007). Set

$$\varphi_n = \varphi_n(\{X_k\}) = \sup\{|P(B\,|\,A) - P(B)|; P(A) > 0, A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_n^{\infty}\},$$

where $\mathcal{F}_{-\infty}^{0} = \sigma(X_k,\ k \le 0)$ and $\mathcal{F}_n^{\infty} = \sigma(X_k,\ k \ge n)$, for each integer $n \ge 1$. This is the uniform strong mixing coefficient introduced by Ibragimov, it satisfies $0 \le \varphi_n \le 1$ (for other properties see Chapter 3, vol. I in Bradley, 2007).

Denote by $\bar{\mu}$ the measure satisfying $\bar{\mu}(B) = \mu(-B)$, for every Borel set B. The main result of this note is the following theorem.

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Theorem. *Let a strictly stationary sequence* $\{X_k\}$ *satisfy* (1.1) *with* μ *non-degenerate and assume that the condition* (1.2) *holds. Then* $\mu \star \bar{\mu}$ *is strictly stable with some exponent* $\alpha \in (0, 2]$ *and if also* $\varphi_1 < 1$ *then, for any* $q \in (0, \alpha)$,

$$E|b_n^{-1} S_n - A_n|^q \to \int_{-\infty}^{\infty} |x|^q \mu(dx) \quad \text{as } n \to \infty. \tag{1.3}$$

This statement generalizes Lemma 4 in Szewczak (2009) (see (2.8)) for dependent random sequences and Theorem 6.1 in de Acosta and Giné (1979) for independent random sequences. The necessary and sufficient conditions for $\{X_k\} \in \mathcal{DA}(\mu)$, under (1.2), can be found in Jakubowski (1993) (see also Jakubowski and Szewczak, 1990, Szewczak, 1992, Szewczak, 1998), therefore the Theorem completes these results with the convergence of moments. For a fuller treatment on domains of attraction for strictly stationary sequences see e.g. pp. 402–403, 427–428, vol. I and pp. 197–200, vol. II in Bradley (2007). As an application of the Theorem recall stable limit theorems for continued fractions in Samur (1989). It is also worth pointing out that in view of Example 2 in Jakubowski and Szewczak (1990) some condition to eliminate the so-called telescoping pathology is unavoidable in the Theorem (here we use $\varphi_1 < 1$). Namely, if we take $\{Y_k\}$ and $\{Z_k\}$ to be i.i.d. symmetric sequences and define 1-dependent sequence $X_k = Y_k + Z_k - Z_{k+1}$, where $x^2 P(|Y_1| > x) \sim 1$ and $xP(|Z_1| > x) \sim 1$ as $x \to \infty$, then of course $\frac{S_n}{\sqrt{n \ln n}}$ converges to the standard normal distribution, but we have no convergence of moments for $q \in [1, 2)$ (see Corollary 1 in Szewczak, 2012). Thus, in view of the Theorem *a fortiori* $\varphi_1(\{X_k\}) = 1$ because the condition (1.2) with $b_n = \sqrt{n \ln n}$ holds for $\{X_k\}$.

2. Proof

We follow the arguments on pp. 91–92 in Araujo and Giné (1980) (see also the proof of Lemma 4 in Szewczak, 2009). Let $\{\hat{X}_k\}$ be an independent copy of $\{X_k\}$ and $\hat{S}_n = \sum_{k=1}^{n} \hat{X}_k$. It is clear that by (1.1) we obtain

$$\mathcal{L}(b_n^{-1}(S_n - \hat{S}_n)) \to \mu \star \bar{\mu} \quad \text{as } n \to \infty.$$

Note that for any complex numbers z_1, z_2, z_3, z_4 of modulus at most 1 we have

$$|z_1 z_2 - z_3 z_4| \le |z_1 - z_3| + |z_2 - z_4|.$$

By this, $\{X_k - \hat{X}_k\}$ satisfies the condition (1.2), whence by Theorem 3.1 in Jakubowski (1993), $\mu \star \bar{\mu}$ is strictly stable with some exponent $\alpha \in (0, 2]$. Thus, for the time being, we assume that $\mathcal{L}(S_n)$ are symmetric and $A_n \equiv 0$. For any $\epsilon > 0$, one can find $t_0 = t_0(\epsilon, \varphi_1)$, $N_0 = N_0(t_0)$ such that $i > N_0$

$$P(b_i^{-1}|S_i| > t_0) \le \mu(|x| > t_0) < \frac{\epsilon}{4}$$

and

$$P(b_i^{-1}|S_j| > t_0) < \frac{\epsilon}{4},$$

for $j = 1, 2, \ldots, N_0 - 1$. Therefore for $n > N_0$

$$\begin{aligned}\max_{1 \le i \le n} P(b_n^{-1}|S_i| > t_0) &\le \max_{1 \le j < N_0} P(b_n^{-1}|S_j| > t_0) + \max_{N_0 \le i \le n} P(b_n^{-1}|S_i| > t_0) \\ &\le \frac{\epsilon}{4} + \max_{N_0 \le i \le n} P(b_i^{-1}|S_i| > t_0) \le \frac{\epsilon}{2}.\end{aligned} \tag{2.4}$$

Recall a formula in lines −5, −4 on p. 69 in Szewczak (2010),

$$P(|S_n| + (m-1)\max_{1 \le i \le n}|X_i| > t) \ge (2 \min_{m \le k \le n} P(2|S_k| \le s) - 1 - \varphi_m) P(\max_{1 \le k \le n-m+1} |S_k| > s + t). \tag{2.5}$$

Since $n > N_0$ and $x > t_0$, by (2.5) (with $m = 1$, $s = 2xb_n$, $t = xb_n$) and (2.4)

$$P\left(\max_{1 \le i \le n}\left|\frac{S_i}{b_n}\right| > 3x\right) \le \frac{1}{1 - \varphi_1 - \epsilon} P\left(\left|\frac{S_n}{b_n}\right| > x\right).$$

Suppose $d > 3t_0$ and $\delta \in (0, (1 - \varphi_1)(1 - \varphi_1 - \epsilon))$ is such that

$$P(b_n^{-1}|S_n| > 3^{-1}d) \le \delta,$$

for $N_0 \le N_\delta \le n$. Consequently, for $n > N_\delta$

$$P\left(\max_{1 \le k \le m}\left|\frac{S_{nk} - S_{n(k-1)}}{b_{mn}}\right| > d\right) \le \frac{1}{1 - \varphi_1 - \epsilon} P\left(\left|\frac{S_{mn}}{b_{mn}}\right| > \frac{d}{3}\right) \le \frac{\delta}{1 - \varphi_1 - \epsilon}.$$