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Two-sided moment estimates for a class of nonnegative chaoses

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ABSTRACT

We derive two-sided bounds for moments of random multi-linear forms (random chaoses) with nonnegative coefficients generated by independent nonnegative random variables X_i which satisfy the following condition on the growth of moments: $\|X_i\|_{2p} \le A\|X_i\|_p$ for any i and $p \ge 1$. The estimates are deterministic and exact up to multiplicative constants which depend only on the order of chaos and the constant A in the moment assumption.

1. Introduction

In this paper we study homogeneous tetrahedral chaoses of order d, i.e. random variables of the form

$$S = \sum_{1 \le i_1,\ldots,i_d \le n} a_{i_1,\ldots,i_d} X_{i_1} \cdot \cdots \cdot X_{i_d},$$

where $X_1, \ldots, X_n$ are independent random variables and $(a_{i_1,\ldots,i_d})$ is a multi-indexed symmetric array of real numbers such that $a_{i_1,\ldots,i_d} = 0$ if $i_l = i_m$ for some $m \ne l, m, l \le d$.

Chaoses of order $d = 1$ are just sums of independent random variables the object quite well understood. Latała (1997) derived two-sided bounds for $\| \sum a_i X_i \|_p$ under general assumptions that either a_i, X_i are nonnegative or X_i are symmetric. The case $d \ge 2$ is much less understood. There are papers presenting two-sided bounds for moments of S in special cases when (X_i) have normal distribution (Latała, 2006), have logarithmically concave tails (Adamczak and Latała, 2012) or logarithmically convex tails (Kolesko and Latała, 2015).

The purpose of this note is to derive two-sided bounds for $\|S\|_p$ if coefficients $(a_{i_1,\ldots,i_d})$ are nonnegative and (X_i) are independent, nonnegative and satisfy the following moment condition for some $k \in \mathbb{N}$,

$$\|X_i\|_{2p} \le 2^k \|X_i\|_p \quad \text{for every } p \ge 1. \tag{1}$$

The main idea is that if a r.v. X_i satisfy (1) then it is comparable with a product of k i.i.d. variables with logarithmically concave tails. In this way the problem reduces to the result of Latała and Łochowski (2003) which gives two-sided bounds for moments of nonnegative chaoses generated by r.v's with logarithmically concave tails.

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2. Notation and main results

We set $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ for a real r.v. Y and $p \geq 1$, $\log(x) = \log_2(x)$ and ln stands for the natural logarithm. By C, t_0 (sometimes $C(k,d), t_0(k,d)$) we denote constants that may depend on k, d and may vary from line to line. We write $A \sim_{k,d} B$ if $A \cdot C(k,d) \geq B$ and $B \cdot C(k,d) \geq A$.

Let $\{X_i^{(1)}\}, \ldots, \{X_i^{(d)}\}$ be independent r.v's. We set

$$N_i^{(r)}(t) = -\ln \mathbb{P}(X_i^{(r)} \geq t).$$

We say that $X_i^{(r)}$ has logarithmically concave tails if the function $N_i^{(r)}$ is convex. We put

$$B_p^{(r)} = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^n N_i^{(r)}(v_i) \leq p \right\}$$

and

$$\|(a_{i_1,\ldots,i_d})\|_p = \sup \left\{ \sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} \prod_{r=1}^d \left(1 + v_{i_r}^{(r)}\right) \mid \left(v_i^{(r)}\right) \in B_p^{(r)} \right\}.$$

We will show the following.

Theorem 2.1. *Let $(X_i^{(r)})_{r \leq d, i \leq n}$ be independent non-negative random variables satisfying (1) and $\mathbb{E}X_i^{(r)} = 1$. Then for any non-negative coefficients $(a_{i_1,\ldots,i_d})_{i_1,\ldots,i_d \leq n}$ we obtain*

$$\frac{1}{C(k,d)} \|(a_{i_1,\ldots,i_d})\|_p \leq \left\| \sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdot \cdots \cdot X_{i_d}^{(d)} \right\|_p \leq C(k,d) \|(a_{i_1,\ldots,i_d})\|_p.$$

Theorem 2.1 in the same way as the proof of Theorem 2.2 in Latała and Łochowski (2003) yields the following two-sided bounds for tails of random chaoses.

Theorem 2.2. *Under the assumptions of Theorem 2.1 there exist constants $0 < c(k,d), C(k,d) < \infty$ depending only on d and k such that for any $t \geq 0$ we have*

$$\mathbb{P}\left(\sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdot \cdots \cdot X_{i_d}^{(d)} \geq C(k,d) \|(a_{i_1,\ldots,i_d})\|_p \right) \leq e^{-p}$$

and

$$\mathbb{P}\left(\sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdot \cdots \cdot X_{i_d}^{(d)} \geq c(k,d) \|(a_{i_1,\ldots,i_d})\|_p \right) \geq \min(c(k,d), e^{-p}).$$

Now we are ready to present two-sided bounds for decoupled chaoses. We define in this case $N_i(t) = -\ln \mathbb{P}(X_i \geq t)$,

$$B_p = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^n N_i(v_i) \leq p \right\}$$

and

$$\|(a_{i_1,\ldots,i_d})\|'_p = \sup \left\{ \sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} \prod_{r=1}^d \left(1 + v_{i_r}^{(r)}\right) \mid \left(v_i^{(r)}\right) \in B_p \right\}.$$

Theorem 2.3. *Let $(X_i)_{i \leq n}$ be nonnegative independent r.v's satisfying (1) and $\mathbb{E}X_i = 1$. Then for any symmetric array of nonnegative coefficients $(a_{i_1,\ldots,i_d})_{i_1,\ldots,i_d \leq n}$ such that*

$$a_{i_1,\ldots,i_d} = 0 \quad \text{if } i_l = i_m \text{ for some } m \neq l,\ m, l \leq d \tag{2}$$

we get

$$\frac{1}{C(k,d)} \|(a_{i_1,\ldots,i_d})\|'_p \leq \left\| \sum_{1 \leq i_1,\ldots,i_d \leq n} a_{i_1,\ldots,i_d} X_{i_1} \cdot \cdots \cdot X_{i_d} \right\|_p \leq C(k,d) \|(a_{i_1,\ldots,i_d})\|'_p.$$