Statistics and Probability Letters 117 (2016) 1–11

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Testing exponentiality of the residual life, based on dynamic cumulative residual entropy

M. Chahkandi *, H. Alizadeh Noughabi

Department of Statistics, University of Birjand, Birjand, Iran

ARTICLE INFO

Article history:
Received 11 February 2015
Received in revised form 1 May 2016
Accepted 1 May 2016
Available online 10 May 2016

Keywords:
Consistent test
Entropy
Kullback–Leibler divergence
Mean residual lifetime function
Monte Carlo simulation

ABSTRACT

A new test statistic is developed for testing exponentiality of the remaining life of a system. Asymptotic properties of the test statistic are discussed. Then, Monte Carlo simulation is used to compare the power of the proposed test with that of an existing test.

1. Introduction

Shannon entropy is a known measure of uncertainly that today has many applications in various fields. For a continuous random variable X with distribution function F and density function f, Shannon entropy is defined as

$$H(f) = E(-\log f(X)) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx.$$

Moreover, Kullback and Leibler (1951) defined a measure of information discrepancy between $g(x)$ and $f(x)$ as

$$KL(g:f) = \int_{-\infty}^{\infty} g(x)\log\frac{g(x)}{f(x)}dx,$$

where g and f are two density functions. $KL(g:f)$ is nonnegative and the equality holds iff $f(x) = g(x)$ almost everywhere. It should be noted that $KL(g:f)$ is not symmetric, so it is not a distance function. It is a measure of directed divergence between g and f.

The KL information can be expressed based on Shannon entropy, and then its estimate can be applied as a goodness-of-fit test statistic, see for example, Arizono and Ohta (1989), Park (2005), Choi and Kim (2006), Balakrishnan et al. (2007) and Habibi Rad et al. (2011). Recently, Rao et al. (2004) developed a new measure of information called cumulative residual entropy (CRE), which is the extension of the entropy to the survival function, as

$$CRE(F) = -\int_{-\infty}^{\infty} \bar{F}(x)\log\bar{F}(x)dx,$$

* Corresponding author.
E-mail addresses: mchahkandi@birjand.ac.ir, ma.chahkandi@yahoo.com (M. Chahkandi).

http://dx.doi.org/10.1016/j.spl.2016.05.001
0167-7152/

where $\bar{F} = 1 - F$ is the survival function of X. Hence, it is worth to extend the KL information to the survival function and verify its relation with CRE to construct a goodness-of-fit test statistic. Baratpour and Habibi Rad (2012) introduced an extension of KL information for a continuous and nonnegative random variable in terms of the survival function, which can be called cumulative residual KL information (CRKL), as

$$CRKL(G:F) = \int_0^\infty \left(\bar{G}(x)\log\frac{\bar{G}(x)}{\bar{F}(x)}\right)dx - [E(Y) - E(X)].$$

Then, they estimated the CRKL and proposed a test of fit for the exponential distribution based on CRKL. Moreover, Baratpour and Habibi Rad (2015) constructed a goodness-of-fit-test for the exponential distribution based on progressively Type II censored data via CRE.

Asadi and Zohrevand (2007) presented the dynamic version of CRE and called it dynamic cumulative residual entropy (DCRE). DCRE is defined as

$$\begin{aligned} DCRE(F;t) &= -\int_t^\infty \bar{F}_t(x)\log\bar{F}_t(x)dx \\ &= -\frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)\log\bar{F}(x)dx + m_F(t)\log\bar{F}(t), \end{aligned}$$

where $\bar{F}_t(x) = \frac{\bar{F}(x)}{\bar{F}(t)}$, and $m_F(t)$ is the mean residual lifetime of random variable X with cdf F, at time t, which is defined as $m_F(t) = E(X - t|X \geq t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$. Note that DCRE is the CRE of random variable $[X - t|X > t]$. Recently, Chamany and Baratpour (2014), extended CRKL information to its dynamic version and called it DCRKL. Chamany and Baratpour (2014) defined DCRKL divergence of two cdfs F and G as

$$\begin{aligned} DCRKL(G:F;t) &= \int_0^\infty \frac{\bar{G}(x+t)}{\bar{G}(t)}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\frac{\bar{G}(t)}{\bar{G}(x+t)} - \log\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\frac{\bar{G}(t)}{\bar{G}(x+t)}\right) - 1\right)dx \\ &= \int_t^\infty \bar{G}_t(x)\left(\frac{\bar{F}_t(x)}{\bar{G}_t(x)} - \log\left(\frac{\bar{F}_t(x)}{\bar{G}_t(x)}\right) - 1\right)dx \\ &= \int_t^\infty \bar{G}_t(x)\log\left(\frac{\bar{G}_t(x)}{\bar{F}_t(x)}\right)dx + (m_F(t) - m_G(t)), \end{aligned} \tag{1}$$

where $m_F(t)$ and $m_G(t)$ are the mean residual lifetimes of X and Y at time t, respectively. Using the same approach in Lemma 2.1 presented by Baratpour and Habibi Rad (2012), we can show that $DCRKL(G:F;t) \geq 0$ and the equality holds if and only if $\bar{F}(x) = \bar{G}(x)$.

It should be mentioned that Ebrahimi and Kirmani (1996) also defined a dynamic version of KL information by comparing the residual life distribution at age $t \geq 0$, as

$$I(F,G;t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}\frac{\bar{G}(t)}{g(x)}dx, \tag{2}$$

where $f_t(x) = \frac{f(x)}{\bar{F}(t)}$. One can see that $DCRKL(G:F;t)$ is a version of $I(F:G;t)$ that using the survival function rather than the density function. Thus, DCRKL can be estimated from sample data and this estimator asymptotically converges to the true value.

This paper considers the problem of testing exponentiality for the residual lifetime of a system. We will find an estimate of $DCRKL(G:F;t)$ and apply it as a goodness-of-fit test statistic and compare its power with the existing test (Ebrahimi and Kirmani, 1996). In Section 2, we first study some extensions of KL information to its dynamic version, and obtain an approximation of DCRKL based on Fisher information. Then, we study the relation between DCRKL and DCRE differences. In Section 3, we follow two approaches, based on maximum DCRE and minimizing DCRKL criterion, for parameter estimation and obtain two test statistics. Park et al. (2012) obtained the same results for CRKL. In Section 4, the critical values of the proposed tests are obtained, and their power values are computed against different alternative models and compared with those of existing test. Finally, in Section 5, the proposed tests are used in a real-life data analysis for illustrative purpose.

2. Preliminary results

In this section, the extension of KL information to its dynamic version is considered. We also obtain an approximation of DCRKL based on Fisher information and then study the relation between DCRKL and DCRE differences.

2.1. Extension of KL information to DCRKL

Some extensions of KL information have been studied by Ebrahimi and Kirmani (1996), Asadi et al. (2004), Baratpour and Habibi Rad (2012), Park et al. (2012) and Park and Shinb (2014). We first consider the direct extension of KL information to