Statistics and Probability Letters 117 (2016) 216–220

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Stochastic comparisons of total capacity of weighted-k-out-of-n systems

Rabi-Allah Rahmani, Muhyiddin Izadi *, Baha-Eldin Khaledi

Department of Statistics, Razi University, Kermanshah, Iran

ARTICLE INFO

Article history:
Received 18 February 2016
Received in revised form 5 June 2016
Accepted 6 June 2016
Available online 14 June 2016

Keywords:
Convolution of independent random variables
Hazard rate order
Increasing failure rate
Mean residual life order and usual stochastic order

ABSTRACT

Stochastic ordering problems in weighted-k-out-of-n systems are considered. The influence of component lifetimes and weights on the total capacity is studied. When it is allowed to allocate lifetimes to the weights, an optimal allocation to attain maximum total capacity is obtained.

1. Introduction

Let $T_1, T_2, \ldots, T_n$ be a set of n non-negative independent random variables representing the component lifetimes of a weighted-k-out-of-n system with the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. It is said that the system works at time t if

$$S_{\mathbf{w}}^{\mathbf{T}}(t) = \sum_{i=1}^{n} w_i I(T_i > t) \geq k,$$

where $S_{\mathbf{w}}^{\mathbf{T}}(t)$ is the total weight (total capacity) of the system at time t. Therefore, the lifetime of the system, denoted by $T_{\mathbf{w}}^{\mathbf{T}}$, is

$$T_{\mathbf{w}}^{\mathbf{T}} = \inf\{t : S_{\mathbf{w}}^{\mathbf{T}}(t) < k\}.$$

Clearly, weighted-k-out-of-n system is reduced to the usual k-out-of-n system when all the components have the same weight equal to 1, i.e., $\mathbf{w} = (1, \ldots, 1)$. As described in Samaniego and Shaked (2008), there are many real systems that can be modelled by weighted-k-out-of-n systems, therefore investigating stochastic properties of various characteristics, such as survival function, mean residual life function and hazard rate function of $S_{\mathbf{w}}^{\mathbf{T}}(t)$ and $T_{\mathbf{w}}^{\mathbf{T}}$ and their behaviour with respect to t as well as $(w_1, \ldots, w_n)$ are of practical importance in the reliability theory and applications.

Wu and Chen (1994) introduced this system and proposed an algorithm to compute its reliability. In the literature, most of the work about weighted-k-out-of-n systems have been focused on the computing of the non-dynamic system reliability. See, for example, Higashiyama (2001), Chen and Yang (2005), Li and Zuo (2008), Samaniego and Shaked (2008), Eryilmaz and Tutuncu (2009), Eryilmaz (2013), Eryilmaz (2014) and Mo et al. (2015). Eryilmaz (2015a) recently studied the mean

* Corresponding author. Fax: +98 83 3 427 4561.
E-mail address: m.izadi@razi.ac.ir (M. Izadi).

http://dx.doi.org/10.1016/j.spl.2016.06.005
0167-7152/

time to failure (MTTF) of weighted-k-out-of-n systems and provided an algorithm to simulate the MTTF. The components importance of weighted-k-out-of-n systems has been studied by Eryilmaz and Bozbulut (2014) and Rahmani et al. (2016).

Eryilmaz and Sarikaya (2014), because of complexity of the stochastic model, obtained a closed form of survival function of $T_{\mathbf{w}}^{\mathbf{T}}$ for the case when the components and weights are only of two types. Eryilmaz (2015b) further studied distribution and mean of two conditional random variables

$$(S_{\mathbf{w}}^{\mathbf{T}}(s)|T_{\mathbf{w}}^{\mathbf{T}} > s) \tag{1.1}$$

and

$$(S_{\mathbf{w}}^{\mathbf{T}}(t) - S_{\mathbf{w}}^{\mathbf{T}}(s)|T_{\mathbf{w}}^{\mathbf{T}} > s), \tag{1.2}$$

for $t < s$. Eryilmaz (2015b) used (1.1) to analyse the residual capacity of the system while it is working at time s and used (1.2) to evaluate the capacity loss between time points t and s. The main purpose of this paper is to study the effect of the component weights and component lifetimes on the total and residual capacity of the weighted-k-out-of-n system.

Next, we recall some definitions of stochastic orderings that are used later in this paper. Throughout this paper, increasing means nondecreasing and decreasing means nonincreasing and we will assume that all expectations exist. Let X and Y be two nonnegative random variables with distribution functions F and G, survival functions $\bar{F}(= 1 - F)$ and $\bar{G}$ and hazard rate functions $r_F(= f/\bar{F})$ and r_G, respectively. The random variable X is said to be smaller than the random variable Y according to the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$, for all x. The random variable X is said to be smaller than the random variable Y according to the hazard rate order (denoted by $X \leq_{hr} Y$) if $r_F(t) \geq r_G(t)$, which is equivalent to that $\frac{\bar{G}(t)}{\bar{F}(t)}$ is increasing in $t > 0$. The random variable X is said to be smaller than the random variable Y according to the mean residual life order (denoted by $X \leq_{mrl} Y$) if $E(X - t|X > t) \leq E(Y - t|Y > t)$, for all $t > 0$.

It is known that the hazard rate order implies both the usual stochastic order and the mean residual life order. For more details on the above notions of stochastic orderings the reader is referred to Shaked and Shanthikumar (2007).

Consider a weighted-k-out-of-n system consisting of n independent components with the component lifetime vector $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ and the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ and another weighted-k-out-of-n system consisting of n independent components with the component lifetime vector $\mathbf{T}' = (T'_1, T'_2, \ldots, T'_n)$ and the weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)$. In Section 2, we show that for $i = 1, \ldots, n$, if either

- $T_i =_{st} T'_i$ and $w_i \leq w'_i$, or
- $T_i \leq_{st} T'_i$ and $w_i = w'_i$, then for all $t > 0$,

$$S_{\mathbf{w}}^{\mathbf{T}}(t) \leq_{hr} S_{\mathbf{w}'}^{\mathbf{T}'}(t).$$

It is proved that if $w_1 \leq w_2 \leq \ldots \leq w_n$ and $T_1 \leq_{st} T_2 \leq_{st} \ldots \leq_{st} T_n$, then

$$S_{\mathbf{w}}^{\boldsymbol{\pi}^{-0}}(t) \leq_{st} S_{\mathbf{w}}^{\boldsymbol{\pi}}(t) \leq_{st} S_{\mathbf{w}}^{\boldsymbol{\pi}^{0}}(t),$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ is any permutation of $\{1, 2, \ldots, n\}$, $\boldsymbol{\pi}^0 = (1, 2, \ldots, n)$, $\boldsymbol{\pi}^{-0} = (n, n-1, \ldots, 1)$ and $S_{\mathbf{w}}^{\boldsymbol{\pi}}(t)$ is the total capacity of a weighted-k-out-of-n system with the component lifetimes $(T_{\pi_1}, \ldots, T_{\pi_n})$ at time t. An application of this result is explained. In Section 2, we also show that the mean residual capacity of a weighted-k-out-of-n system is decreasing over the time.

2. Comparison of total capacity and mean residual capacity of weighted-*k*-out-of-*n* systems

We use the following results to prove the main results in this section. The results might be of independent interest.

Lemma 2.1. *Let T and T' be two non-negative random variables, and let a and b be two positive constants. For all $t > 0$,*

(i) *$aI(T > t)$ is IFR.*
(ii) *If $T \leq_{st} T'$, then $aI(T > t) \leq_{hr} aI(T' > t)$.*
(iii) *$aI(T > t) \leq_{hr} bI(T > t)$ for $a \leq b$.*

The following theorem shows how the component lifetimes and their weights affect the total capacity of a weighted-k-out-of-n system.

Theorem 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a random vector of independent non-negative random variables representing the component lifetimes of a weighted-k-out-of-n system with weight vector $\mathbf{w} = (w_1, \ldots, w_n)$ and let $\mathbf{T}' = (T'_1, \ldots, T'_n)$ be a random vector of independent non negative random variables representing the component lifetimes of another weighted-k-out-of-n system with weight vector $\mathbf{w}' = (w'_1, \ldots, w'_n)$. Then, for all $t > 0$,*

(a) $\mathbf{w} \leq \mathbf{w}' \Rightarrow S_{\mathbf{w}}^{\mathbf{T}}(t) \leq_{hr} S_{\mathbf{w}'}^{\mathbf{T}}(t)$ *and*
(b) $(T_i \leq_{st} T'_i,\ i = 1, \ldots, n) \Rightarrow S_{\mathbf{w}}^{\mathbf{T}}(t) \leq_{hr} S_{\mathbf{w}}^{\mathbf{T}'}(t)$.
(c) *For $t_1 \leq t_2$, $S_{\mathbf{w}}^{\mathbf{T}}(t_1) \geq_{hr} S_{\mathbf{w}}^{\mathbf{T}}(t_2)$.*