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A characterization of the normal distribution by the independence of a pair of random vectors

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ABSTRACT

Kagan and Shalaevski (1967) have shown that if the random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ are i.i.d. and the distribution of $\sum_{i=1}^n (\mathbb{X}_i + a_i)^2 a_i \in \mathbb{R}$ depends only on $\sum_{i=1}^n a_i^2$, then each $\mathbb{X}_i \sim N(0, \sigma)$. In this paper, we will give other characterizations of the normal distribution which are formulated in a similar spirit.

1. Introduction

It will be shown that the formulae are much simplified by the use of cumulative moment functions, or semi-invariants, in place of the crude moments (Fisher, 1929).

The original motivation for this paper comes from a desire to understand the results about characterization of normal distribution which were shown in Cook (1971) and Kagan and Shalaevski (1967). They proved, that the characterizations of a normal law are given by a certain invariance of the noncentral chi-square distribution. It is a known fact that if $\mathbb{X}_1, \dots, \mathbb{X}_n$ are i.i.d. and following the normal distribution $N(0, \sigma)$ then the distribution of the statistic $\sum_{i=1}^n (\mathbb{X}_i + a_i)^2$, $a_i \in \mathbb{R}$ depends on $\sum_{i=1}^n a_i^2$ only (see Bryc, 1995; Moran, 1968). Kagan and Shalaevski (1967) have shown that if the random variables $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n$ are independent and identically distributed and the distribution of $\sum_{i=1}^n (\mathbb{X}_i + a_i)^2$ depends only on $\sum_{i=1}^n a_i^2$, then each $\mathbb{X}_i$ is normally distributed as $N(0, \sigma)$. Cook generalized this result replacing independence of all $\mathbb{X}_i$ by the independence of $(\mathbb{X}_1, \dots, \mathbb{X}_m)$ and $(\mathbb{X}_{m+1}, \dots, \mathbb{X}_n)$ and removing the requirement that $\mathbb{X}_i$ have the same distribution. The theorem proved below gives a new look on this subject, i.e. we will show that in the statistic $\sum_{i=1}^n (\mathbb{X}_i + a_i)^2 = \sum_{i=1}^n \mathbb{X}_i^2 + 2\sum_{i=1}^n \mathbb{X}_i a_i + \sum_{i=1}^n a_i^2$ only the linear part $\sum_{i=1}^n \mathbb{X}_i a_i$ is important. In particular, from the above result we get Cook Theorem from Cook (1971), but under the assumption that all moments exist. Note that Cook does not assume any moments, but he gets this result under integrability assumptions imposed on the corresponding random variable. This paper is removing or at least relaxing its integrability assumptions.

The paper is organized as follows. In Section 2 we review basic facts about cumulants. Next in Section 3 we state and prove the main results. In this section we also discuss the problem.

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2. Cumulants and moments

Cumulants were first defined and studied by the Danish scientist T. N. Thiele. He called them semi-invariants. The importance of cumulants comes from the observation that many properties of random variables can be better represented by cumulants than by moments. We refer to Brillinger (1975) and Gnedenko and Kolmogorov (1954) for further detailed probabilistic aspects of this topic.

Given a random variable $\mathbb{X}$ with the moment generating function $g(t)$, its ith cumulant r_i is defined as

$$r_i(\mathbb{X}) := r_i(\underbrace{\mathbb{X}, \dots, \mathbb{X}}_{i\text{-times}}) = \frac{d^i}{dt^i}\Big|_{t=0} \log(g(t)).$$

That is,

$$\sum_{i=0}^{\infty} \frac{m_i}{i!} t^i = g(t) = \exp\Big(\sum_{i=1}^{\infty} \frac{r_i}{i!} t^i\Big)$$

where m_i is the ith moment of $\mathbb{X}$.

Generally, if σ denotes the standard deviation, then

$$r_1 = m_1, \qquad r_2 = m_2 - m_1^2 = \sigma, \qquad r_3 = m_3 - 3m_2 m_1 + 2m_1^3.$$

The joint cumulant of several random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ of order $(i_1, \dots, i_n)$, where i_j are nonnegative integers, is defined by a similar generating function $g(t_1, \dots, t_n) = E\big(e^{\sum_{i=1}^n t_i \mathbb{X}_i}\big)$

$$r_{i_1+\cdots+i_n}(\underbrace{\mathbb{X}_1, \dots, \mathbb{X}_1}_{i_1\text{-times}}, \dots, \underbrace{\mathbb{X}_n, \dots, \mathbb{X}_n}_{i_n\text{-times}}) = \frac{d^{i_1+\cdots+i_n}}{dt_1^{i_1} \dots dt_n^{i_n}}\Big|_{t=0} \log(g(t_1, \dots, t_n)),$$

where $t = (t_1, \dots, t_n)$.

Random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ are independent if and only if, for every $n \geq 1$ and every non-constant choice of $\mathbb{Y}_i \in \{\mathbb{X}_1, \dots, \mathbb{X}_n\}$, where $i \in \{1, \dots, k\}$ (for some positive integer $k \geq 2$) we get $r_k(\mathbb{Y}_1, \dots, \mathbb{Y}_k) = 0$.

Cumulants of some important and familiar random distributions are listed as follows:

- The Gaussian distribution $N(\mu, \sigma)$ possesses the simplest list of cumulants: $r_1 = \mu$, $r_2 = \sigma$ and $r_n = 0$ for $n \geq 3$,
- for the Poisson distribution with mean λ we have $r_n = \lambda$.

These classical examples clearly demonstrate the simplicity and efficiency of cumulants for describing random variables. Apparently, it is not accidental that cumulants encode the most important information of the associated random variables. The underlying reason may well reside in the following four important properties (which are in fact related to each other):

- (Translation Invariance) For any constant c, $r_1(\mathbb{X} + c) = c + r_1(\mathbb{X})$ and $r_n(\mathbb{X} + c) = r_n(\mathbb{X})$, $n \geq 2$.
- (Additivity) Let $\mathbb{X}_1, \dots, \mathbb{X}_m$ be any independent random variables. Then, $r_n(\mathbb{X}_1 + \cdots + \mathbb{X}_m) = r_n(\mathbb{X}_1) + \cdots + r_n(\mathbb{X}_m)$, $n \geq 1$.
- (Commutative property) $r_n(\mathbb{X}_1, \dots, \mathbb{X}_n) = r_n(\mathbb{X}_{\sigma(1)}, \dots, \mathbb{X}_{\sigma(n)})$ for any permutation $\sigma \in S_n$.
- (Multilinearity) r_k are the k-linear maps.

For more details about cumulants and probability theory, the reader can consult Lehner (2004) or Rota and Jianhong (2000).

3. The characterization theorem

The main result of this paper is the following characterization of normal distribution in terms of independent random vectors.

Theorem 3.1. *Suppose vectors $(\mathbb{S}_1, \mathbb{Y})$ and $(\mathbb{S}_2, \mathbb{Z})$ with all moments are independent and $\mathbb{S}_1, \mathbb{S}_2$ are nondegenerate. If for every $a, b \in \mathbb{R}$ the linear combination $a\mathbb{S}_1 + \mathbb{Y} + b\mathbb{S}_2 + \mathbb{Z}$ has the law that depends on (a, b) through $a^2 + b^2$ only, then random variables $\mathbb{S}_1, \mathbb{S}_2$ have the same normal distribution and $\operatorname{cov}(\mathbb{S}_1, \mathbb{Y}) = \operatorname{cov}(\mathbb{S}_2, \mathbb{Z}) = 0$.*

Remark 3.2. This result is well known if $\mathbb{S}_1$, $\mathbb{Y}$, $\mathbb{S}_2$ and $\mathbb{Z}$ are mutually independent. In order to understand what is the extent of the contribution, we give an example where the assumptions of the theorem are true (i.e. $\mathbb{S}_1$, $\mathbb{Y}$ are not independent and $\mathbb{S}_2$, $\mathbb{Z}$ are not independent).

Example 1. Let $\mathbb{S}_1, \mathbb{S}_2$ be independent and identically distributed standard normal random variables and let $\mathbb{Y} = \mathbb{S}_1^2$, $\mathbb{Z} = \mathbb{S}_2^2$, then

$$a\mathbb{S}_1 + \mathbb{Y} + b\mathbb{S}_2 + \mathbb{Z} = (\mathbb{S}_1 + a/2)^2 + (\mathbb{S}_2 + b/2)^2 - \frac{1}{4} \times (a^2 + b^2).$$

It is a known fact that the distribution of the statistic $(\mathbb{S}_1 + a/2)^2 + (\mathbb{S}_2 + b/2)^2$ depends on $a^2 + b^2$ only (see introduction). This means that the distribution of $a\mathbb{S}_1 + \mathbb{Y} + b\mathbb{S}_2 + \mathbb{Z}$ depends only on $a^2 + b^2$.