Statistics and Probability Letters 112 (2016) 1–4

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Conditional independence and conditioned limit laws

Ioannis Papastathopoulos

School of Mathematics and Maxwell Institute, University of Edinburgh, Edinburgh, EH9 3FD, United Kingdom

ARTICLE INFO

Article history:
Received 4 August 2015
Received in revised form 28 December 2015
Accepted 29 December 2015
Available online 14 January 2016

MSC:
primary 60GXX
secondary 60G70

Keywords:
Asymptotic independence
Conditional independence
Conditioned limit laws
Random norming

ABSTRACT

Conditioned limit laws constitute an important and well developed framework of extreme value theory that describe a broad range of extremal dependence forms including asymptotic independence. We explore the assumption of conditional independence of X_1 and X_2 given X_0 and study its implication in the limiting distribution of (X_1, X_2) conditionally on X_0 being large. We show that under random norming, conditional independence is always preserved in the conditioned limit law but might fail to do so when the normalisation does not include the precise value of the random variable in the conditioning event.

1. Introduction

Extreme value theory is a highly active area of research and its methods and applications are the epitome of risk modelling and statistical estimation of rare events. Extreme events that occur in a broad range of disciplines such as in environmental processes or in finance and insurance are typically multivariate in nature and usually have tremendous socio-economic impact. The recent technological advances have resulted in an ever-increasing amount of information available across the whole spectrum of applied sciences. As such, when modelling data in several dimensions, one is typically confronted by the *curse of dimensionality*. It is widely recognised that the construction of more efficient statistical models and techniques that overcome this problem is imperative. *Conditional independence* constitutes one of the most fundamental tools and concepts in this direction (Besag, 1974; Dawid, 1979; Whittaker, 1990; Lauritzen, 1996; Cox and Wermuth, 1996). On the other hand, the central concept of regular variation, its extensions and refinements, provide the recipe for the development of asymptotically justified extreme value models. The purpose of this short note is to illustrate some implications of conditional independence in conditioned limit laws (Heffernan and Tawn, 2004; Heffernan and Resnick, 2007), a key and well-developed framework that embodies a broad range of extremal dependence forms.

For ease of exposition, consider a random vector (X_0, X_1, X_2) in $\mathbb{R}^3$. The result developed in this paper extends to higher-dimensional settings for X_1 and X_2 in a straightforward manner so this is not restrictive. We assume that X_1 and X_2 are conditionally independent given X_0. Informally, this means that the conditional distribution of X_1 given (X_0, X_2) is equal to the conditional distribution of X_1 given X_0 alone. In other words, once X_0 is known any further information about X_2 is irrelevant to uncertainty about X_1 and therefore it readily follows that for any $x_1, x_2 \in \mathbb{R}$,

$$\mathbb{P}\left(X_1 < x_1, X_2 < x_2 \mid X_0\right) = \mathbb{P}\left(X_1 < x_1 \mid X_0\right) \mathbb{P}\left(X_2 < x_2 \mid X_0\right), \tag{1}$$

E-mail address: i.papastathopoulos@ed.ac.uk.

http://dx.doi.org/10.1016/j.spl.2015.12.028
0167-7152/

almost surely. We show that conditional independence may naturally be *preserved* in limiting laws of random vectors given an extreme component (Heffernan and Tawn, 2004; Heffernan and Resnick, 2007). Conditioned limit laws provide a rich description of extremes of random vectors that do not necessarily grow at the same rate and may exhibit *asymptotic independence* which means that the coefficient

$$\chi = \lim_{p \to 1-} \mathbb{P}\left[F_{X_1}(X_1) > p, F_{X_2}(X_2) > p \mid F_{X_0}(X_0) > p\right], \tag{2}$$

can be 0. Conditioned limit laws were systematically studied for first time by Heffernan and Tawn (2004) who examined the limiting conditional distribution of affinely transformed random vectors as the conditioning variable becomes large. Assuming identical margins with distribution function being asymptotically equivalent to the unit exponential distribution, i.e., $F_{X_0}(x) = F_{X_1}(x) = F_{X_2}(x) \sim 1 - \exp(-x)$, as $x \to \infty$, Heffernan and Tawn showed that for a broad range of dependence structures of (X_0, X_1, X_2), there exist scaling functions $\alpha_1, \alpha_2 : (0, \infty) \mapsto (0, \infty)$, location functions $\beta_1, \beta_2 : (0, \infty) \mapsto \mathbb{R}$ and a joint distribution G on $[-\infty, \infty) \times [-\infty, \infty)$ with non-degenerate marginals, such that as $t \to \infty$

$$\mathbb{P}\left[\frac{X_1 - \beta_1(X_0)}{\alpha_1(X_0)} < x_1, \frac{X_2 - \beta_2(X_0)}{\alpha_2(X_0)} < x_2 \,\middle|\, X_0 > t\right] \xrightarrow{\mathcal{D}} G(x_1, x_2), \tag{3}$$

on $[-\infty, \infty] \times [-\infty, \infty]$ where $\xrightarrow{\mathcal{D}}$ stands for weak convergence. Although the original formulation of Heffernan and Tawn relied on the existence of densities, formulation (3) is presented here in the compact form of Heffernan and Resnick (2007) who provided a formal mathematical examination of the more general conditioned limit formulation that there exists a joint distribution H on $[-\infty, \infty) \times [-\infty, \infty)$ with non-degenerate marginals such that as $t \to \infty$

$$\mathbb{P}\left[\frac{X_1 - \beta_1(t)}{\alpha_1(t)} < x_1, \frac{X_2 - \beta_2(t)}{\alpha_2(t)} < x_2 \,\middle|\, X_0 > t\right] \xrightarrow{\mathcal{D}} H(x_1, x_2), \tag{4}$$

on $[-\infty, \infty] \times [-\infty, \infty]$, subject to the sole assumption of X_0 belonging to the domain of attraction of an extreme value distribution. Expressions (3) and (4) can be rephrased more generally as special cases of joint probability convergence; here we use the conditional representation to highlight the connection with conditional independence.

Limit expressions (3) and (4) differ in the way X_1 and X_2 are normalised since in expression (3), the precise value of X_0 that occurs with $X_0 > t$ is used, whereas in expression (4) only partial information about the value of X_0 is exploited since only the level value t that X_0 exceeds is used. Following the terminology of Heffernan and Resnick (2007) we refer to limit expressions (3) and (4) as the conditional extreme value model with *random norming* and *deterministic norming*, respectively, and for the remaining part we assume without loss of generality that X_0 has a unit Pareto distribution, i.e., $\mathbb{P}(X_0 < x) = 1 - 1/x, x > 1$. The unit Pareto marginal scale for X_0 is primarily chosen for convenience but the result of this paper can be stated, with modified proofs (Kulik and Soulier, 2015), for X_0 being in the domain of attraction of an extreme value distribution. We also assume that the two limit expressions (3) and (4) simultaneously hold for the same pair of norming functions. In general, this may not always be true but a necessary and sufficient condition is to assume that $(\alpha_i, \beta_i), i = 1, 2$, in expression (4) are extended regularly varying, i.e., there exist $\rho_1, \rho_2, \kappa_1, \kappa_2 \in \mathbb{R}$ such that as $t \to \infty$,

$$\frac{\alpha_i(tx)}{\alpha_i(t)} \to x^{\rho_i} \quad \text{and} \quad \frac{\beta_i(tx) - \beta_i(t)}{\alpha_i(t)} \to \psi_i(x),$$

for $x > 0$, where

$$\psi_i(x) = \begin{cases} \kappa_i\,(x^{\rho_i} - 1)\,/\rho_i & \rho_i \neq 0 \\ \kappa_i \log x & \rho = 0, \end{cases} \tag{5}$$

for $i = 1, 2$ (Resnick and Zeber, 2014).

In Section 2 we state the main theorem of the paper which shows that the conditional independence property (1) is preserved in the conditional extreme value model with random norming, meaning that for any $x_1, x_2 \in \mathbb{R}$,

$$G(x_1, x_2) = G_1(x_1)\, G_2(x_2), \tag{6}$$

where $G_1(x_1) = \lim_{x_2 \to \infty} G(x_1, x_2)$ and $G_2(x_2) = \lim_{x_1 \to \infty} G(x_1, x_2)$, but might fail to do so in the conditional extreme value model with deterministic norming. In Section 3 we discuss practical consequences of conditional independence in conditioned limit laws. A proof is given in the Appendix.

2. Main result

Let $\pi_1, \pi_2 : (0, \infty) \times \mathcal{B}(\bar{\mathbb{R}}) \mapsto [0, 1]$ be Markov kernels defined by

$$\pi_1(x, A) = \mathbb{P}(X_1 \in A \mid X_0 = x) \quad \text{and} \quad \pi_2(x, A) = \mathbb{P}(X_2 \in A \mid X_0 = x).$$

Let also $\sigma(X_0) = \{{X_0}^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$ be the sigma algebra generated by X_0.