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Asymptotics of powers of binomial and multinomial probabilities

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ABSTRACT

Fix positive integers $k \geq 2, j \geq 2$ and numbers $p_1, p_2, \ldots, p_k$ such that $0 < p_i < 1$ for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} p_i = 1$. For a positive integer n, let

$$b_{n,j,k}(p_1, p_2, \ldots, p_k) \equiv \sum_{(n_1, n_2, \ldots, n_k) \in T_{n,k}} \left(\frac{n!}{n_1! n_2! \cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \right)^j,$$

where $T_{n,k}$ is the set $\left\{ (n_1, n_2, \ldots, n_k) : n_i \in \{0, 1, 2, \ldots, n\}, \sum_{i=1}^{k} n_i = n \right\}$. Then there exists $0 < b_{j,k}(p_1, p_2, \ldots, p_k) < \infty$ such that

$$n^{(j-1)(k-1)/2} b_{n,j,k}(p_1, p_2, \ldots, p_k) \to b_{j,k}(p_1, p_2, \ldots, p_k) \tag{1}$$

as $n \to \infty$.

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1. Introduction

Pólya and Szegő (1970, problem 40, p. 42) have shown that for any positive integers j and $n, j \geq 2$,

$$\sum_{k=0}^{n} \binom{n}{k}^j \sim \left(2^n \sqrt{\frac{2}{\pi n}} \right)^j \sqrt{\frac{\pi n}{2j}} \tag{2}$$

in the sense that the ratio of the two sides goes to 1 as $n \to \infty$ (for each fixed j). A proof of this using the central limit theorem is given in Farmer and Leth (2005).

There are several possible generalizations suggested by (2). Here are two of them.

1. Let $0 < p < 1$ and $q = 1 - p$. For any positive integers $n \geq 1$ and $j \geq 2$, let

$$a_{n,j}(p) = \sum_{k=0}^{n} \left(\binom{n}{k} p^k q^{n-k} \right)^j. \tag{3}$$

For fixed j and p, how does $a_{n,j}(p)$ behave as $n \to \infty$?

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2. Let $0 < p_i < 1, i = 1, 2, \dots, k$ with $\sum_{i=1}^k p_i = 1$ and $k \geq 3$. Fix $j \geq 2$ and let

$$b_{n,j,k}(p_1, \dots, p_k) = \sum_{(n_1,n_2,\dots,n_k)\in T_{n,k}} \left(\frac{n!}{n_1!n_2!\cdots n_k!} p_1^{n_1} \cdots p_k^{n_k} \right)^j, \tag{4}$$

where $T_{n,k}$ is the set $\left\{(n_1, \dots, n_k) \ : \ n_i \text{ is a nonnegative integer for } 1 \leq i \leq k, \text{ and } \sum_1^k n_i = n\right\}$. For each fixed $j, k, p_1, \dots, p_k$, how does $b_{n,j,k}(p_1, \dots, p_k)$ behave as $n \to \infty$?

2. The binomial case

Indeed, if $j = 2$ and $p = 1/2$, (2) provides the answer to question 1 above. A natural open question is to extend Pólya and Szegő's result in (2) to powers $j \geq 2$ and probabilities $p \in (0, 1), p \neq 1/2$. Such an extension is given in Theorem 2.1. Our proof uses the Fourier inversion formula for convergent l_1 sequences on Euclidean lattices and a local central limit theorem from probability theory.

Theorem 2.1. *Fix $0 < p < 1$ and let $q = 1 - p$. For any positive integer $j \geq 2$,*

$$n^{(j-1)/2} a_{n,j}(p) \to j^{-1/2} \left(2\pi p(1-p)\right)^{-(j-1)/2} \quad \text{as } n \to \infty, \tag{5}$$

where $a_{n,j}(p)$ is given in Eq. (3).

Proof. First, let $j = 2$ and let X_n and Y_n be two independent random variables with binomial (n, p) distribution. That is, for all $0 \leq k, s \leq n$, k, s and n nonnegative integers,

$$P\left(X_n = k, Y_n = s\right) = \binom{n}{k} p^k q^{n-k} \binom{n}{s} p^s q^{n-s}.$$

Then

$$a_{n,2}(p) = P\left(X_n = Y_n\right) = \sum_{k=0}^n P\left(X_n = k\right) P\left(Y_n = k\right) = P\left(X_n - Y_n = 0\right).$$

Now $X_n - Y_n$ can be written as $\sum_{i=0}^n \left(\delta_i' - \delta_i''\right)$, where $\left\{\delta_i', \delta_i'', i = 1, \dots, n\right\}$ are independent Bernoulli random variables with distribution $P\left(\delta_i' = 1\right) = p$ and $P\left(\delta_i' = 0\right) = q = 1 - p$. If $\delta_i \equiv \delta_i' - \delta_i''$, $i = 1, 2, \dots, n$, then $\{\delta_i\}_{i=1}^n$ are independent and identically distributed random variables with distribution

$$\begin{aligned} P\left(\delta_i = 0\right) &= P\left(\delta_i' = 0 = \delta_i''\right) + P\left(\delta_i' = 1 = \delta_i''\right) = p^2 + (1-p)^2, \\ P\left(\delta_i = 1\right) &= P\left(\delta_i' = 1, \delta_i'' = 0\right) = p\left(1-p\right), \\ P\left(\delta_i = -1\right) &= P\left(\delta_i' = 0, \delta_i'' = 1\right) = p\left(1-p\right). \end{aligned}$$

Further, $X_n - Y_n \equiv \sum_{i=1}^n \delta_i$ for all $n \geq 1$. Also, for $\theta \in \mathbb{R}$, $i = \sqrt{-1}$,

$$\begin{aligned} E\left(e^{i\theta\delta_1}\right) &= p^2 + (1-p)^2 + p(1-p)\left(e^{i\theta} + e^{-i\theta}\right) \\ &= p^2 + (1-p)^2 + 2p(1-p) - 2p(1-p)(1-\cos\theta) \\ &= 1 - 2p(1-p)(1-\cos\theta). \end{aligned}$$

This implies, by independence of δ_i, $i = 1, \dots, n$, that

$$E\left(e^{i\theta(X_n - Y_n)}\right) = E\left(e^{i\theta\sum_{j=1}^n \delta_j}\right) = \left(1 - 2p(1-p)(1-\cos\theta)\right)^n.$$

Also, by the Fourier inversion formula for probability distributions on the integers (Feller, 1968),

$$P\left(X_n - Y_n = 0\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - 2p(1-p)(1-\cos t)\right)^n dt.$$

Thus,

$$\sqrt{n} P\left(X_n - Y_n = 0\right) = \frac{1}{2\pi} \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left(1 - 2p(1-p)\left(1 - \cos\frac{u}{\sqrt{n}}\right)\right)^n du, \tag{6}$$