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Standard maximum likelihood drift parameter estimator in the homogeneous diffusion model is always strongly consistent

Yuliya Mishura

Taras Shevchenko National University of Kyiv, Mechanics and Mathematics Faculty, Volodymyrska 60, 01601 Kyiv, Ukraine

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ABSTRACT

We consider the homogeneous stochastic differential equation with unknown parameter to be estimated. We prove that the standard maximum likelihood estimate is strongly consistent under very mild conditions. The conditions for strong consistency of the discretized estimator are established as well.

1. Introduction

There is an extended literature devoted to standard and nonstandard approaches to the drift parameter estimation in the diffusion models, both for discrete and continuous observations. We mention only the books Liptser and Shiryaev (2001); Prakasa Rao (1987); Heyde (1997); Kessler et al. (2012) and references therein. Many complicated models have been studied. However, there was a curious gap even in the case of simplest homogeneous diffusion model: there were no conditions for the strong consistency of the standard maximum likelihood estimator that are close to be necessary and are sufficiently mild. We have filled the gap, applying the results of the paper of Mijatovic and Urusov (2012) and have proved that, in some sense, the standard maximum likelihood estimator is always strongly consistent unless the drift coefficient is identically zero.

The paper is organized as follows. In Section 2 some preliminaries are given. In Section 3 we prove that the denominator in the stochastic representation of the maximum likelihood estimator tends to infinity with probability 1 under very mild conditions and deduce from here the strong consistency. In Section 4 we establish the sufficient conditions for the strong consistency of the discretized version of the maximum likelihood estimator. Some simulation results and discussion on the possible generalizations are included in Section 5.

E-mail addresses: yumishura@gmail.com, myus@univ.kiev.ua.

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2. Preliminaries

Let $(\Omega, \Im, \{\Im_t\}_{t\geq 0}, P)$ be a complete probability space with filtration that satisfies the standard conditions. Let $W = \{W_t, \Im_t, t \geq 0\}$ be a standard Wiener process. Consider a homogeneous diffusion process $X = \{X_t, \Im_t, t \geq 0\}$ that is a solution to the stochastic differential equation

$$X_t = x_0 + \theta \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s. \tag{2.1}$$

Here $x_0 \in \mathbb{R}$, $\theta \in \mathbb{R}$ is the unknown parameter to be estimated, $a, b : \mathbb{R} \to \mathbb{R}$ are measurable functions, $b(x) \neq 0$ for any $x \in \mathbb{R}$, and a is not zero identically. In general, we only need the existence and uniqueness of the weak solution of Eq. (2.1) on the whole semi-axis. Recall that any of the following groups of conditions on a and b supplies the existence–uniqueness for the strong solution:

Yamada conditions (Yamada and Watanabe, 1971; Ikeda and Watanabe, 1989):

(i) Linear growth: there exists $K > 0$ such that for any $x \in \mathbb{R}$

$|a(x)| + |b(x)| \leq K(1 + |x|)$;

(ii) There exists a convex increasing function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $k(0) = 0$, $\int_{0+} k^{-1}(u)du = +\infty$ and for any $x, y \in \mathbb{R}$ $|a(x) - a(y)| \leq k(|x - y|)$;

(iii) There exists a strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(0) = 0$, $\int_{0+} \rho^{-2}(u)du = +\infty$ and for any $x, y \in \mathbb{R}$ $|b(x) - b(y)| \leq \rho(|x - y|)$.

Krylov–Zvonkin conditions (Krylov and Zvonkin, 1981):

(i) Function a is bounded, function b is separated from 0: $b(x) \geq \alpha > 0$, $x \in \mathbb{R}$;

(ii) Function b has locally bounded variation: for any $N > 0$

$$var_{[-N,N]}b := \sup_{-N=x_0<x_1<\cdots<x_n=N} \sum |b(x_{k+1}) - b(x_k)| < \infty.$$

Existence of the weak solution of Eq. (2.1) holds under continuity and linear growth of the coefficients. This was initially proved in Skorokhod (1965). The conditions of existence and uniqueness of the weak solution were then generalized in Krylov (2009) and the most general conditions were obtained in Engelbert and Schmidt (1981, 1989, 1991).

3. Strong consistency of the drift parameter maximum-likelihood estimator constructed for continuous observations

Denote the functions $c(x) = \frac{a(x)}{b^2(x)}$, $d(x) = \frac{a^2(x)}{b^2(x)}$. In what follows we suppose that the following condition holds:

(A) functions $\dfrac{1}{b^2}$, d and $\dfrac{d}{b^2}$ are locally integrable.

Furthermore, denote $L_t(x) = b^2(x) \lim_{\varepsilon\downarrow\infty} \frac{1}{2\varepsilon} \int_0^t 1\{|X_s - x| < \varepsilon\}ds$ the local time of the process X at the point x on the interval $[0, t]$, $t \geq 0$. Then, according, e.g., to Pitman and Yor (2003), for any locally integrable function f the following equality holds:

$$\int_0^t f(X_s)ds = \int_{\mathbb{R}} \frac{f(x)}{b^2(x)} L_t(x)dx, \quad t \geq 0.$$

Therefore, under the condition of local integrability, $\int_0^t d(X_s)ds < \infty$ for any $t > 0$. As is well known, the likelihood function for Eq. (2.1) has the form

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{\theta \int_0^t \frac{a(X_s)}{b(X_s)}dW_s + \frac{\theta^2}{2}\int_0^t d(X_s)ds\right\} = \exp\left\{\theta \int_0^t c(X_s)dX_s - \frac{\theta^2}{2}\int_0^t d(X_s)ds\right\},$$

and the maximum likelihood estimator of parameter θ constructed by the observations of X on the interval $[0, t]$, has the form

$$\hat{\theta}_t = \frac{\int_0^t c(X_s)dX_s}{\int_0^t d(X_s)ds} = \theta + \frac{\int_0^t \frac{a(X_s)}{b(X_s)}dW_s}{\int_0^t d(X_s)ds}. \tag{3.1}$$

In order to establish the criteria of the strong consistency of $\hat{\theta}_t$ in terms of the coefficients a and b, denote $\varphi(x) = \exp\left\{-2\theta \int_0^x c(y)dy\right\}$, $\Phi(x) = \int_0^x \varphi(y)dy$. Concerning the asymptotic behavior of the integral $\int_0^t d(X_s)ds$ under the fixed value of parameter $\theta \neq 0$, two cases can be considered.