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Isotonicity properties of generalized quantiles

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ABSTRACT

We investigate whether several families of generalized quantiles (expectiles, L^p-quantiles and M-quantiles) respect various stochastic orders (the usual stochastic order, the convexity order, and the p-convexity orders).

We employ techniques from monotone comparative statics developed by Topkis (1978) and Milgrom and Shannon (1994), in order to provide sufficient as well as necessary conditions for isotonicity.

We show that expectiles with $\alpha \leq 1/2$ are basically the only generalized quantiles that are isotonic with respect to the $\leq_{icv}$ ordering; more generally, the L^p-quantiles are isotonic with respect to the p-convex order.

1. Introduction

It is well known that the quantiles of a distribution can be defined as the minimizers of an asymmetric piecewise linear loss function:

$$x^*_{\alpha,X} = \arg\min_{x \in R} \pi_\alpha(X, x) \tag{1}$$

with

$$\pi_\alpha(X, x) := E[\alpha(X - x)^+ + (1 - \alpha)(X - x)^-]. \tag{2}$$

This very important property has been exploited in various areas of statistics. It has been introduced in the context of statistical decision theory (see Ferguson, 1967); it lies at the heart of the quantile regression approach developed by Koenker and Bassett (see Koenker, 2005); more recently, it has been employed in financial risk management by Rockafellar and Uryasev (2002).

Alternative specifications of the loss function (2) have originated several kind of generalized quantiles: Newey and Powell (1987) considered a quadratic loss (expectiles), Chen (1996) a p-power loss (L^p-quantiles), Breckling and Chambers (1988) generic convex losses (M-quantiles). Similar functionals have also been considered in the actuarial literature under the name of Orlicz quantiles by Bellini and Rosazza Gianin (2012).

When considering the usual quantiles, it is well known that if $X \leq_{st} Y$, then $x^*_{\alpha,X} \leq x^*_{\alpha,Y}$; that is, quantiles are isotonic with respect to the usual stochastic order $\leq_{st}$. In this paper we investigate similar isotonicity properties for generalized quantiles; this is relevant in that it provides additional information on their significance and possible interpretation.

For example, we will show that the expectiles $\left(\text{with } \alpha \leq \frac{1}{2}\right)$ are isotone with respect to the $\leq_{icv}$ ordering, also known as second order stochastic dominance in the financial literature. This property shows that expectiles might be reasonable risk

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measures for a risk averse agent, as it was suggested also by Muller (2010). Moreover, we will show that the expectiles are basically the only M-quantiles that share this isotonicity property.

To prove these results, we employ techniques from monotone comparative statics, developed in the seminal papers of Topkis (1978) and Milgrom and Shannon (1994). Roughly, this theory provides sufficient and necessary conditions for increasing optimal solutions in a parametric optimization problem as (1) (the variable X is seen as a parameter), under suitable conditions on the loss function $\pi_\alpha(X, x)$.

2. Expectiles

The expectiles have been introduced by Newey and Powell (1987) as the minimizers

$$x^*_{\alpha,X} := \arg\min_{x\in R} \pi_\alpha(X, x)$$

of the asymmetric quadratic loss

$$\pi_\alpha(X, x) = E[\alpha((X-x)^+)^2 + (1-\alpha)((X-x)^-)^2],$$

with $\alpha \in (0, 1)$ and $E[|X|^2] < +\infty$. In order to avoid unnecessarily heavy notations, we will write simply x^* or x^*_α when no possibility of confusion arises. Clearly if $\alpha = \frac{1}{2}$ then

$$\pi_{\frac{1}{2}}(X, x) = \frac{1}{2}E[(X-x)^2]$$

so that

$$x^*_{\frac{1}{2},X} = E[X].$$

Newey and Powell proved that x^* is always unique and satisfies the first order condition

$$x^* - E[X] = \frac{2\alpha-1}{1-\alpha}\int_{x^*}^{+\infty}(t-x^*)dF(t) \tag{3}$$

that can be rewritten in the equivalent forms

$$(1-\alpha)\int_{-\infty}^{x^*}F(t)dt = \alpha\int_{x^*}^{+\infty}\overline{F}(t)dt \tag{4}$$

or

$$\alpha = \frac{E[(X-x^*)^-]}{E[|X-x^*|]}, \tag{5}$$

with $\overline{F}(t) = 1 - F(t)$. These equations are well defined and have a unique solution also for $X \in L^1$. The basic properties of expectiles are summarized in the following proposition.

Proposition 1 (*Newey and Powell, 1987*)**.** *Let $E[|X|] < +\infty$, $\alpha \in (0, 1)$, $x^*_{\alpha,X}$ as in (3). It follows that:*

- *$x^*_{\alpha,X}$ is strictly monotonic with respect to α*
- *$x^*_{\alpha,X}$ is positively homogeneous: $x^*_{\alpha,\lambda X} = \lambda x^*_{\alpha,X}$, $\forall\lambda \geq 0$*
- *$x^*_{\alpha,X}$ is translation equivariant: $x^*_{\alpha,X+c} = x^*_{\alpha,X} + c$, $\forall c \in R$*
- *if X is symmetric with respect to x_0, then*

$$\frac{x^*_{\alpha,X} + x^*_{1-\alpha,X}}{2} = x_0$$

- *if X has a continuous density, then $x^*_{\alpha,X}$ is C^1 as a function of α, with*

$$\frac{dx^*_\alpha}{d\alpha} = \frac{E[|X-x^*|]}{(1-\alpha)F(x^*) + \alpha\overline{F}(x^*)}.$$

Typically expectiles are more concentrated around the mean than the corresponding quantiles. However, a general comparison result is still lacking and Koenker (1992) showed that for the infinite variance distribution function given by

$$F(t) = \begin{cases} \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{4+t^2}}\right), & t \geq 0 \\ \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{4+t^2}}\right), & t < 0 \end{cases}$$

expectiles coincide with quantiles.