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Identifying the spectral representation of Hilbertian time series

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ABSTRACT

We provide $\sqrt{n}$-consistency results regarding estimation of the spectral representation of covariance operators of Hilbertian time series, in a setting with imperfect measurements. This is a generalization of the method developed in Bathia et al. (2010). The generalization relies on an important property of centered random elements in a separable Hilbert space, namely, that they lie almost surely in the closed linear span of the associated covariance operator. We provide a straightforward proof to this fact. This result is, to our knowledge, overlooked in the literature. It incidentally gives a rigorous formulation of Principal Component Analysis in Hilbert spaces.

1. Introduction

In this paper, we provide theoretical results regarding estimation of the spectral representation of the covariance operator of stationary Hilbertian time series. This is a generalization of the method developed in Bathia et al. (2010) to a setting of random elements in a separable Hilbert space. The approach taken in Bathia et al. (2010) relates to functional Principal Component Analysis and, similarly to the latter, relies strongly on the Karhunen–Loéve (K–L) theorem. The authors develop the theory in the context of curve time series, with each random curve in the sequence satisfying the conditions of the K–L Theorem which, together with a stationarity assumption, ensures that the curves can all be expanded in the same basis—namely, the basis induced by their zero-lag covariance function. The idea is to identify the dimension of the space M spanned by this basis (finite by assumption), and to estimate M, when the curves are observed with some degree of error. Specifically, it is assumed that the statistician can only observe the curve time series (Y_t), where

$$Y_t = X_t + \epsilon_t,$$

whereas the curve time series of interest is actually (X_t). Here Y_t, X_t and ϵ_t are random functions (curves) defined on $[0, 1]$. Estimation of M in this framework was previously addressed in Hall and Vial (2006) assuming the curves are i.i.d. (in t), a setting in which the problem is indeed unsolvable in the sense that one cannot separate X_t from ϵ_t. The authors propose a Deus ex machina solution which consists in assuming that ϵ_t goes to 0 as the sample size grows. Bathia et al. (2010) in turn

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resolve this issue by imposing a dependence structure in the evolution of (X_t). Their key assumption is that, at some lag k, the kth lag autocovariance matrix of the random vector composed by the Fourier coefficients of X_t in M, is full rank. In our setting this corresponds to Assumption (A1) (see below).

In Bathia et al. (2010) it is assumed that each of the stochastic processes $(X_t(u) : u \in [0, 1])$ satisfy the conditions of the K–L Theorem (and similarly for ϵ_t), and as a consequence the curves are in fact random elements with values in the Hilbert space $L^2[0, 1]$. Therefore, since every separable Hilbert space is isomorphic to $L^2[0, 1]$, the idea of a generalization to separable Hilbert spaces of the aforementioned methodology might seem, at first, rather dull. The issue is that *transforming the data (that is, applying the isomorphism) may not be feasible nor desirable in applications*. For instance, the isomorphism may involve calculating the Fourier coefficients in some 'rule-of-thumb' basis which might yield infinite series even when the curves are actually finite dimensional.

The approach that we take here relies instead on the key feature that a centered Hilbertian random element of strong second order, lies almost surely in the closed linear span of its corresponding covariance operator. Understanding of this statement requires a bit of theory which we now shortly review. Given a separable Hilbert space H endowed with inner-product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, a *random element* in H is a Borel measurable map $\xi : \Omega \to H$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space. Also, for $q \geq 1$, if $\mathbb{E}\|\xi\|^q < \infty$ we say that ξ is of *strong order* q and write $\xi \in L^q_{\mathbb{P}}(H)$. In this case, there is a unique element $h_\xi \in H$ satisfying the identity $\mathbb{E}\langle \xi, f \rangle = \langle h_\xi, f \rangle$ for all $f \in H$. The element h_ξ is called the *expectation* of ξ and is denoted by $\mathbb{E}\xi$. If $\mathbb{E}\xi = 0$ we say that ξ is *centered*. If ξ and η are centered random elements in H of strong order 2, they are said to be (mutually) strongly orthogonal if, for each $h, f \in H$, it holds that $\mathbb{E}\langle h, \xi \rangle \langle f, \eta \rangle = 0$. Further, let $\mathcal{L}(H)$ denote the Banach space of bounded linear operators acting on H, and let $A \in \mathcal{L}(H)$. If for some (and hence, all) orthonormal basis (e_j) of H one has $\|A\|_2 := \sum_{j=1}^{\infty} \|A(e_j)\|^2 < \infty$, we say that A is a *Hilbert–Schmidt operator*. The set $\mathcal{L}_2(H)$ of Hilbert–Schmidt operators is itself a separable Hilbert space with inner-product $\langle A, B \rangle_2 = \sum_{j=1}^{\infty} \langle A(e_j), B(e_j) \rangle$, with $\|\cdot\|_2$ being the induced norm. An operator $T \in \mathcal{L}(H)$ is said to be *nuclear*, or *trace-class*, if $T = AB$ for some Hilbert–Schmidt operators A and B. If $\xi \in L^2_{\mathbb{P}}(H)$, its *covariance operator* is the nuclear operator $R_\xi(h) := \mathbb{E}\langle \xi, h \rangle \xi$, $h \in H$. More generally, if $\xi, \eta \in L^2_{\mathbb{P}}(H)$, their *cross-covariance* operator is defined, for $h \in H$, by $R_{\xi,\eta}(h) := \mathbb{E}\langle \xi, h \rangle \eta$.

The key result that we mentioned above allows one to dispense with considerations of 'sample path properties' of a random curve by addressing the spectral representation of a Hilbertian random element directly. In other words, the Karhunen–Loéve Theorem is just a special case[1] of a more general phenomenon. The formal statement of this result (which motivates – and for that matter, justifies – our approach) is not a new one: it appears, for example, in a slightly different guise as an exercise in Vakhania et al. (1987). However, it is in our opinion rather overlooked in the literature, perhaps because of the technical level required for stating it in full generality. Here we provide a statement requiring less theory, and we give a straightforward proof which is, to our knowledge, a new one. In this paper H is always assumed to be a real Hilbert space, but with minor adaptations all the results hold for complex H.

Theorem 1. *Let H be a separable Hilbert space, and assume ξ is a centered random element in H of strong second order, with covariance operator R. Then $\xi \perp \ker(R)$ almost surely.*

Corollary 1. *In the conditions of Theorem 1, let $(\lambda_j : j \in J)$ be the (possibly finite) nonincreasing sequence of nonzero eigenvalues of R, repeated according to multiplicity, and let $\{\varphi_j : j \in J\}$ denote the orthonormal set of associated eigenvectors. Then*

(i) $\xi(\omega) = \sum_{j \in J} \langle \xi(\omega), \varphi_j \rangle \varphi_j$ *in H, almost surely;*
(ii) $\xi = \sum_{j \in J} \langle \xi, \varphi_j \rangle \varphi_j$ *in $L^2_{\mathbb{P}}(H)$.*

Moreover, the scalar random variables $\langle \xi, \varphi_i \rangle$ and $\langle \xi, \varphi_j \rangle$ are uncorrelated if $i \neq j$, with $\mathbb{E}\langle \xi, \varphi_j \rangle^2 = \lambda_j$.

Remark. (a) Although it is beyond the scope of this work, we call attention to the fact that Theorem 1 and Corollary 1 provide a rigorous justification of Principal Component Analysis for Hilbertian random elements. (b) In Corollary 1 either $J = \mathbb{N}$ or, whenever R is of rank $d < \infty$, $J = \{1, \dots, d\}$.

Proofs to the above and subsequent statements are given in the Appendix. We can now adapt the methodology of Bathia et al. (2010) to a more general setting.

2. The model

In what follows $(\Omega, \mathscr{F}, \mathbb{P})$ is a fixed complete probability space. Consider a stationary process $(\xi_t : t \in \mathbb{T})$ of random elements with values in a separable Hilbert space H. Here $\mathbb{T}$ is either $\mathbb{N} \cup \{0\}$ or $\mathbb{Z}$. We assume throughout that ξ_0 is a centered random element in H of strong second order. Of course, the stationarity assumption ensures that these properties are shared by all the ξ_t. Now let

$$R_k(h) := \mathbb{E}\langle \xi_0, h \rangle \xi_k, \quad h \in H,$$

[1] This is not entirely true since the Karhunen–Loéve Theorem states uniform (in $[0, 1]$) $L^2(\Omega)$ convergence.