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On some spectral properties of large block Laplacian random matrices

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ABSTRACT

In this paper, we investigate the spectral properties of the large block Laplacian random matrices when the blocks are general rectangular matrices. Under some moment assumptions of the underlying distributions, we study the convergence of the empirical spectral distribution (ESD) of the large block Laplacian random matrices.

1. Introduction and main results

The study of Laplacian matrices is of special interest for random graphs. Although there are many matrices for a given graph, the most studied are its adjacency matrices and Laplacian matrices. The spectra of the Laplacian matrices are closely related to the properties of the random graphs. See Fiedler (1973), Ding and Jiang (2010) and Jiang (2012) for details. The Laplacian matrices are also called Markov matrices in other areas, one can see Bordenave (2008), Bryc et al. (2006), Bordenave and Lelarge (2010).

In this paper, we are interested in the convergence of the empirical spectral distribution (ESD) of the large block Laplacian matrices with rectangular blocks. Define a $n \times n$ symmetric block random matrices $\mathbf{S}_n$ as following:

$$\mathbf{S}_n = \sum_{k,l=1}^{d} \mathbf{E}_{kl} \otimes \mathbf{G}^{(k,l)},$$

where the $\otimes$ denotes the Kronecker product of matrices, $\mathbf{E}_{kl}$ are the elementary $d \times d$ matrices having 1 at (k, l) and 0 elsewhere, $\{\mathbf{G}^{(k,l)}, 1 \le k \le l \le d\}$ are $n_k \times n_l$ independent rectangular random matrices which we call blocks, and $\mathbf{G}^{(l,k)} = \mathbf{G}^{(k,l)^T}$ for all $k, l = 1, \ldots, d$. It is clear that $n_1 + \cdots + n_d = n$. Let $\{g_{rs}^{(k,l)}, r = 1, \ldots, n_k, s = 1, \ldots, n_l\}$ denote

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the entries of the matrix $\mathbf{G}^{(k,l)}$. For convenience, we write $\mathbf{S}_n = (s_{ij}) = (g_{rs}^{(k,l)})$, the block Laplacian random matrix $\mathbf{M}_n$ is defined by

$$\mathbf{M}_n = \mathbf{S}_n - \mathbf{D}_n, \tag{1.1}$$

where $\mathbf{D}_n = diag(\sum_{j=1}^{n} s_{ij})_{1\le i\le n}$ is a diagonal matrix. The block Laplacian random matrix $\mathbf{M}_n$ can be written as

$$\mathbf{M}_n = \sum_{k,l=1}^{d} \mathbf{E}_{kl} \otimes \mathbf{O}^{(k,l)},$$

where $\mathbf{O}^{(k,l)} = \mathbf{G}^{(k,l)}$ for $k \neq l$ and $\mathbf{O}^{(k,k)}$ is the matrix $\mathbf{G}^{(k,k)}$ with diagonal entries $g_{rr}^{(k,k)}$ being replaced by $-\sum_{l\neq k}\sum_{s=1}^{n_l} g_{rs}^{(k,l)} - \sum_{s\neq r}^{n_k} g_{rs}^{(k,k)}$ for $r = 1, \ldots, n_k$.

Random graph with given expected degree sequence is an important class of random graphs in graph theory. For random graph with some special given expected degree sequences, its corresponding Laplacian matrix is the block Laplacian matrix $\mathbf{M}_n$. The spectral properties of $\mathbf{M}_n$ are substantial to such random graphs. For example, the eigenvalues of the Laplacian matrices are related to the number of spanning trees of the graphs and the algebraic connectivity of the graphs.

The present paper is devoted to the problem of the convergence of the ESD of $\mathbf{M}_n = \mathbf{S}_n - \mathbf{D}_n$. The convergence of the ESD of $\mathbf{S}_n$ is investigated in Ding (2014). The spectral properties of other large block random matrices have been extensively studied because of its applications to graph theory, physics, wireless communications and biology. For some recent developments of the large block random matrices, one can see Basu et al. (2012), Bolla (2004), Dette and Reuther (2010), Far et al. (2008), Li et al. (2011), Oraby (2007).

Now we present our main results. Given any $n \times n$ symmetric matrix $\mathbf{B}$, we use $\lambda_1^{\mathbf{B}} \ge \cdots \ge \lambda_n^{\mathbf{B}}$ to denote its eigenvalues and

$$F^{\mathbf{B}}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{\lambda_i^{\mathbf{B}} \le x\}}$$

for the empirical spectral distribution (ESD) of $\mathbf{B}$.

Theorem 1. *Assume that the entries $(s_{ij}) = (g_{sr}^{(l,k)})$ of the random matrices $\mathbf{S}_n$ satisfy the following assumptions:*

(a) $g_{rs}^{(k,l)} = g_{sr}^{(l,k)}$ *for all* $r = 1, \ldots, n_k, s = 1, \ldots, n_l, k, l = 1 \ldots, d, n_k/n \to \alpha_k \in [0, 1], k = 1, \ldots, d$;
(b) $\{g_{rs}^{(k,l)}, 1 \le r \le n_k, 1 \le s \le n_l\}$ *are independent random variables with mean zero and* $E|g_{rs}^{(k,l)}|^2 = \sigma_{kl}^2 < \infty$ *for* $1 \le k < l \le d$, $\{g_{rs}^{(k,k)}, 1 \le r \le s \le n_k\}$ *are independent random variables with mean zero and* $E|g_{rs}^{(k,k)}|^2 = \sigma_{kk}^2 < \infty$;
(c) $\sup_{r,s,k,l} E|g_{rs}^{(k,l)}|^{4+\delta} < \infty$ *for some* $\delta > 0$.

Then, with probability one, the ESD $F^{(1/\sqrt{n})\mathbf{M}_n}$ of $(1/\sqrt{n})\mathbf{M}_n$ converges weakly to a probability distribution F called the limiting spectral distribution (LSD) of $(1/\sqrt{n})\mathbf{M}_n$.

Remark 1. In Theorem 1, we do not require the entries in the same block are identically distributed.

Theorem 2. *Under the assumptions in Theorem 1, $(1/\sqrt{n})\mathbf{M}_n$ and $(1/\sqrt{n})\mathbf{W}_n = (1/\sqrt{n})(\mathbf{X}_n + \mathbf{Z}_n)$ have the same LSD, where $\mathbf{X}_n$ and $\mathbf{Z}_n$ are independent random matrices with normal entries, $\mathbf{X}_n$ has the same block structure and the same first two moments of all entries as $\mathbf{S}_n$, $\mathbf{Z}_n = diag(z_{ii})_{1\le i\le n}$ with $Ez_{ii} = 0$, $E(z_{ii})^2 = \sum_{l\neq k} n_l\sigma_{kl}^2 + n_k\sigma_{kk}^2$ for $n_1 + \cdots + n_{k-1} < i \le n_1 + \cdots + n_k$.*

It is difficult to obtain the explicit expression of the probability distribution F in Theorem 1. When all σ_{kl}^2 are equal, Bryc et al. (2006) find that the F is the free convolution between a semicircular law and a normal distribution by adopting the free probability theory. If σ_{kl}^2 are not all equal, by Theorem 2, we only need to study the LSD of $(1/\sqrt{n})(\mathbf{X}_n + \mathbf{Z}_n)$ in order to investigate F. By the operator-valued free probability approach, Far et al. (2008) established a system of equations which can be solved numerically to compute the desired LSD of $(1/\sqrt{n})\mathbf{X}_n$. And it is easy to verify that the LSD of $(1/\sqrt{n})\mathbf{Z}_n$ is a normal distribution. Since σ_{kl}^2 are not all equal, both $\mathbf{X}_n$ and $\mathbf{Z}_n$ are not orthogonally invariant, the LSD of $(1/\sqrt{n})(\mathbf{X}_n + \mathbf{Z}_n)$ cannot be computed by free probability theory, and the LSD of $(1/\sqrt{n})(\mathbf{X}_n + \mathbf{Z}_n)$ has not been explicitly given so far. Thus it is a interesting problem for further research to investigate the LSD of $(1/\sqrt{n})(\mathbf{X}_n + \mathbf{Z}_n)$ and $(1/\sqrt{n})\mathbf{M}_n$.

2. Proof of main results

For any two distribution functions F and G, the Lévy distance between them is defined by

$$L(F, G) = \inf\{\varepsilon : F(x - \varepsilon) - \varepsilon \le G(x) \le F(x + \varepsilon) + \varepsilon, \text{ for all } x\}.$$

For a sequence of probability distributions, the convergence in metric L implies convergence in distribution. For the empirical spectral distribution of $n \times n$ real symmetric matrices $\mathbf{A}$, $\mathbf{B}$, by Corollary A.41 in Bai and Silverstein (2010), we know that

$$L^3(F^{\mathbf{A}}, F^{\mathbf{B}}) \le \frac{1}{n} tr(\mathbf{A} - \mathbf{B})^2. \tag{2.2}$$