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A class of random deviation theorems for sums of nonnegative stochastic sequence and strong law of large numbers

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Abstract

In this paper, the notion of limit log-likelihood ratio of random sequences, as a measure of *dissimilarity* between the true density $p_n(x_1, \ldots, x_n)$ $(n = 1, 2, \ldots)$ and the product of their marginals $\prod_{i=1}^{n} p_i(x_i)$, is introduced. Establish a.s. convergence supermartingale by means of constructing new probability density functions and under suitable restrict conditions, some random deviation theorems for arbitrary stochastically dominated continuous random variables and some strong law of large numbers are obtained.

MSC: 60F15; 60B12

Keywords: Random variable; Limit log-likelihood ratio; Random deviation theorem; Stochastically dominated random sequence

1. Introduction

In recent years, some results have been made in the field of deviation of the arithmetic means of random variables. The main problem of the research area going back to Liu (1990, 1997) is to determine a relationship between the true probability distribution and its marginals. Liu (1997) discussed the strong deviation theorems for discrete random variables by using the generating function method, the present paper focuses on the study of the random deviation theorems and strong law of large numbers for sums of arbitrary continuous random variables in more general settings.

Throughout this paper, $\{X_n, n \geqslant 1\}$ will designate $\mathscr{F}$-measurable nonnegative random variables where $\{\mathscr{F}_n, n \geqslant 0\}$ is an increasing sequence of sub-σ-algebras of the basic σ-algebra $\mathscr{F}$ of the underlying probability space $(\Omega, \mathscr{F}, P)$ and $\mathscr{F}_0$ is the trivial σ-algebra $\{\Phi, \Omega\}$. Assume that the joint density functions of $\{X_n, n \geqslant 1\}$ are

$$p_n(x_1, \ldots, x_n), \quad 1 \leqslant i \leqslant n, \ \ n = 1, 2, \ldots \tag{1.1}$$

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and $p_i(x_i)$, $i=1,2,\ldots$ are the marginal density functions of them, let

$$\pi_n(x_1,\ldots,x_n)=\prod_{i=1}^{n}p_i(x_i), \quad n\geqslant 1 \tag{1.2}$$

and let $F_k(x)=P(X_k<x)$, resp.

Definition 1. Let $\{X_n, n\geqslant 1\}$ be a sequence of random variables with joint density function (1.1), and $p_k(x_k)$ be defined as before, $\{\sigma_n, n\geqslant 1\}$ be a sequence of positive integer-valued real numbers with $\sigma_n \uparrow \infty$ as $n\to\infty$, let

$$L_n(\omega)=\begin{cases}\dfrac{\pi_n(X_1,X_2,\ldots,X_n)}{p_n(X_1,X_2,\ldots,X_n)} & \text{if the denominator} > 0,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\gamma_n(\omega)=\ln L_n(\omega) \tag{1.3}$$

with $\ln 0=-\infty$, where ω is a sample point, and X_k stands for $X_k(\omega)$ for brevity.

The random variable

$$\gamma(\omega)=-\liminf_{n}\frac{\gamma_{\sigma_n}(\omega)}{\sigma_n} \tag{1.4}$$

is called the limit log-likelihood ratio, relative to the product of marginal distribution of (1.2), of $X_n, n\geqslant 1$, and it will be shown in (2.19) that $\gamma(\omega)\geqslant 0$, a.s. in any case.

Although $\gamma(\omega)$ is not a proper metric between probability measures, we nevertheless think of it as a measure of "dissimilarity" between their joint distribution $p_n(x_1,x_2,\ldots,x_n)$ and the product $\pi_n(x_1,x_2,\ldots,x_n)$ of their marginals.

Obviously, $\gamma_n(\omega)\equiv 0$, a.s. $(n\geqslant 1)$ if and only if $\{X_n, n\geqslant 1\}$ are independent.

A stochastic process of fundamental importance in the theory of testing hypotheses is the sequence of likelihood ratio. In view of the above discussion of the limit log-likelihood ratio, it is natural to think of $\gamma(\omega)$ as a measure how far (the random deviation) of $\{X_n, n\geqslant 1\}$ is from being independent, how dependent they are. The smaller $\gamma(\omega)$ is, the smaller the deviation is Liu (1997).

Finally, some definitions and preliminary results will be presented prior to established the main results and it will be shown that they play central roles in the proofs.

Definition 2. Let $\{X_n, n\geqslant 1\}$ and $p_k(x)$ be defined by (1.1) and (1.2), the moment transformation and tail probability moment transformation as follows:

$$M_k(s)=\int_0^{\infty}s^x p_k(x)\,\mathrm{d}x, \quad s>0 \tag{1.5}$$

and

$$W_k(s)=\int_0^{\infty}s^x\int_x^{\infty}p_k(t)\,\mathrm{d}t\,\mathrm{d}x, \quad s>0 \tag{1.6}$$

Definition 3. Let $\{X_n, n\geqslant 1\}$ be a sequence of nonnegative random variables, and is said to be:

(1) Stochastically dominated by a nonnegative random variable X (we write $\{X_n, n\geqslant 1\}\prec X$) if there exists a constant $C>0$ such that

$$\sup_{n\geqslant 1}P\{X_n>x\}\leqslant CP\{X>x\} \quad \text{for all } x>0. \tag{1.7}$$

(2) Stochastically dominated in Cesàro sense by a nonnegative random variable X (we write $\{X_n, n\geqslant 1\}\prec X(C)$) if there exists a constant $C>0$ such that

$$\sup_{n\geqslant 1}n^{-1}\sum_{k=1}^{n}P\{X_k>x\}\leqslant CP\{X>x\} \quad \text{for all } x>0. \tag{1.8}$$