Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 126 (2016) 800–831
stochastic processes and their applications
www.elsevier.com/locate/spa

Scaling limits for the exclusion process with a slow site

Tertuliano Franco[a,*], Patrícia Gonçalves[b,c], Gunter M. Schütz[d]

[a] UFBA, Instituto de Matemática, Campus de Ondina, Av. Adhemar de Barros, S/N. CEP 40170-110, Salvador, Brazil
[b] Departamento de Matemática, PUC-RIO, Rua Marquês de São Vicente, no. 225, 22453-900, Rio de Janeiro, RJ, Brazil
[c] CMAT, Centro de Matemática da Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal
[d] Institute of Complex Systems II, Forschungszentrum Jülich, 52428 Jülich, Germany

Received 7 May 2015; received in revised form 27 August 2015; accepted 29 September 2015
Available online 13 October 2015

Abstract

We consider the symmetric simple exclusion processes with a slow site in the discrete torus with n sites. In this model, particles perform nearest-neighbor symmetric random walks with jump rates everywhere equal to one, except at one particular site, *the slow site*, where the jump rate of entering that site is equal to one, but the jump rate of leaving that site is given by a parameter $g(n)$. Two cases are treated, namely $g(n) = 1 + o(1)$, and $g(n) = \alpha n^{-\beta}$ with $\beta > 1$, $\alpha > 0$. In the former, both the hydrodynamic behavior and equilibrium fluctuations are driven by the heat equation (with periodic boundary conditions when in finite volume). In the latter, they are driven by the heat equation with Neumann boundary conditions. We therefore establish the existence of a dynamical phase transition. The critical behavior remains open.

MSC: 60K35; 26A24; 35K55

Keywords: Exclusion processes; Hydrodynamics; Fluctuations; Phase transition; Ornstein–Uhlenbeck process

1. Introduction

In the seventies, Dobrushin and Spitzer, see [22] and references therein, initiated the idea of obtaining a mathematically precise understanding of the emergence of macroscopic behavior

* Corresponding author.
E-mail addresses: tertu@ufba.br (T. Franco), patricia@mat.puc-rio.br (P. Gonçalves), g.schuetz@fz-juelich.de (G.M. Schütz).

http://dx.doi.org/10.1016/j.spa.2015.09.019

in gases or fluids from the microscopic interaction of a large number of identical particles with stochastic dynamics. This approach has turned out to be extremely fruitful both in probability theory and statistical physics (e.g. see the books [24,13]) and it still raises attention nowadays. In this context, recent studies have been made in hydrodynamic limit/fluctuations of interacting particle systems in random/non homogeneous medium, see for instance [9,6,8,12] and references therein.

So far, most of the work done in this field concerns the bulk hydrodynamics, i.e., the derivation of macroscopic partial differential equations arising from the *bulk interactions* of the underlying particle system. To this end, one usually considers an infinite system or a finite torus with periodic boundary conditions and then takes the thermodynamic limit. However, in applications to physical systems one is usually confronted with finite systems, which requires the study of a partial differential equation on a finite interval with prescribed boundary conditions. This raises the question from which microscopic *boundary interactions* a given type of boundary condition emerges at the macroscopic scale.

This is an important issue both for boundary-driven open systems, where boundary interactions can induce long-range correlations [23] and bulk phase transitions due to the absence of particle conservation at the boundaries [3], and for bulk-driven conservative systems on the torus where even a single defect bond between two neighboring sites can change bulk relaxation behavior or lead to macroscopic discontinuities in the hydrostatic density profiles.[1] Given such rich behavior due to boundary effects in non-conservative or bulk-driven systems it is natural to explore the macroscopic role of a microscopic defect on a torus in a conservative system in the *absence* of bulk-driving and to ask whether such a defect can be described on macroscopic scale in terms of a boundary condition for the PDE describing the bulk hydrodynamics.

In this work we address this problem for the symmetric simple exclusion process (SSEP) on the discrete torus in the presence of a defect site. The model can be described as follows. Each site of the discrete torus with n sites, that we denote by $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$, is allowed to have at most one particle. To each site is associated a Poisson clock, all of them being independent. If there is a particle in the associated site, this particle chooses one of its nearest neighbors with equal probability when the clock rings. If the chosen site is empty, the particle jumps to it. Otherwise nothing happens. All sites have a Poisson clock of parameter two, except the origin, which has a Poisson clock of parameter $2g(n)$. If $g(n) < 1$, the origin behaves as a *trap*, and (in average) it keeps a particle there for a longer time than the other sites do. We call this site a *slow site*. The main results of the present work are the hydrodynamic limit and the equilibrium fluctuations for the exclusion process with such a slow site.

Specifically, for $g(n) = 1 + o(1)$ it is shown here that the limit for the time trajectory of the spatial density of particles is given by the solution of the heat equation with periodic boundary conditions, namely:

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u), & t \geq 0,\ u \in \mathbb{T}, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is the one-dimensional continuous torus.

Moreover, considering the same particle system evolving on $\mathbb{Z}$, we prove that the equilibrium fluctuations of the system are driven by a generalized Ornstein–Uhlenbeck process $\mathcal{Y}_t$ which is

[1] See [11,20,1,4,2,17] for numerical, exact and rigorous results for the asymmetric simple exclusion process and [21,19] for a review, including experimental applications of interacting particle systems with boundary interactions in physical and biological systems.