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Approximating the value functions for stochastic differential games with the ones having bounded second derivatives

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Abstract

We show a method of uniform approximation of the value functions of uniformly nondegenerate stochastic differential games in smooth domains up to a constant over K with the ones having second-order derivatives bounded by a constant times K for any $K \geq 1$.

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1. Introduction

Let $\mathbb{R}^d = \{x = (x^1, \dots, x^d)\}$ be a d-dimensional Euclidean space and let $d_1 \geq d$ be an integer. Assume that we are given separable metric spaces A and B, and let, for each $\alpha \in A$, $\beta \in B$, the following functions on $\mathbb{R}^d$ be given:

(i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(x) = (\sigma^{\alpha\beta}_{ij}(x))$,

(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(x) = (b^{\alpha\beta}_i(x))$, and

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(iii) real-valued functions $c^{\alpha\beta}(x) \geq 0$, $f^{\alpha\beta}(x)$, and $g(x)$.

Under natural assumptions which will be specified later one associates with these objects and a bounded domain $G \subset \mathbb{R}^d$ a stochastic differential game with the diffusion term $\sigma^{\alpha\beta}(x)$, drift term $b^{\alpha\beta}(x)$, discount rate $c^{\alpha\beta}(x)$, running cost $f^{\alpha\beta}(x)$, and the final cost $g(x)$ paid when the underlying process first exits from G.

After the order of players is specified in a certain way it turns out (see, for instance, [1,3,13] or Remark 2.2 in [4]) that the value function $v(x)$ of this differential game is a unique continuous in $\bar{G}$ viscosity solution of the Isaacs equation

$$H[v] = 0$$

in G with boundary condition $v = g$ on ∂G, where for a sufficiently smooth function $u = u(x)$

$$L^{\alpha\beta}u(x) := a^{\alpha\beta}_{ij}(x)D_{ij}u(x) + b^{\alpha\beta}_i(x)D_i u(x) - c^{\alpha\beta}(x)u(x),$$

$$a^{\alpha\beta}(x) := (1/2)\sigma^{\alpha\beta}(x)(\sigma^{\alpha\beta}(x))^*, \qquad D_i = \partial/\partial x^i, \qquad D_{ij} = D_i D_j,$$

$$H[u](x) = \sup_{\alpha\in A}\inf_{\beta\in B}[L^{\alpha\beta}u(x) + f^{\alpha\beta}(x)]. \tag{1.1}$$

Under some assumptions one explicitly constructs a convex positive-homogeneous of degree one function $P(u_{ij}, u_i, u)$ such that for any $K \geq 1$ the equation

$$\max(H[u], P[u] - K) = 0 \tag{1.2}$$

in G with boundary condition $u = g$ on ∂G has a unique solution v_K in class $C^{1,1}_{\text{loc}}(G) \cap C(\bar{G})$ with the second-order derivatives bounded by a constant times K divided by the distance to the boundary. Here

$$P[u](x) = P(D_{ij}u(x), D_i u(x), u(x)).$$

The goal of this article is to prove the conjecture stated in [9]: $|v - v_K| \leq N/K$ in G for $K \geq 1$, where N is independent of K. Such a result even in a much weaker form was already used in numerical approximations of solutions of the Isaacs equations in [5].

The result belongs to the theory of partial differential equations. However, the proof we give is purely probabilistic and quite nontrivial involving, in particular, a reduction of differential games in domains to the ones on a smooth manifolds without boundary. The main idea underlying this reduction is explained in the last two sections of [7] and, of course, we represent v_K also as a value function for a corresponding stochastic differential game. Still it is worth mentioning that the methods of the theory of partial differential equations can be used to obtain results similar to ours albeit not that sharp in what concerns the rate of approximations even though for Isaacs equations with much less regular coefficients than ours (see [6]).

The article is organized as follows. Section 2 contains our main result. In Section 3 we prove the dynamic programming principle for stochastic differential games in the whole space. In Section 4 we show how to reduce the stochastic differential game in a domain to the one in the whole space having four more dimensions. Actually, the resulting stochastic differential games lives on a closed manifold without boundary. In Section 5 we prove our main result, Theorem 2.2.

2. Main result

We start with our assumptions.