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Continuous-time limit of repeated interactions for a system in a confining potential ☆

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Abstract

We study the continuous-time limit of a class of Markov chains coming from the evolution of classical open systems undergoing repeated interactions. This repeated interaction model has been initially developed for dissipative quantum systems in Attal and Pautrat (2006) and was recently set up for the first time in Deschamps (2012) for classical dynamics. It was particularly shown in the latter that this scheme furnishes a new kind of Markovian evolutions based on Hamilton's equations of motion. The system is also proved to evolve in the continuous-time limit with a stochastic differential equation. We here extend the convergence of the evolution of the system to more general dynamics, that is, to more general Hamiltonians and probability measures in the definition of the model. We also present a natural way to directly renormalize the initial Hamiltonian in order to obtain the relevant process in a study of the continuous-time limit. Then, even if Hamilton's equations have no explicit solution in general, we obtain some bounds on the dynamics allowing us to prove the convergence in law of the Markov chain on the system to the solution of a stochastic differential equation, via the infinitesimal generators.

Keywords: Stochastic differential equations; Relativistic diffusion processes; Infinitesimal generators; Classical open systems; Hamiltonian systems; Repeated interactions

1. Introduction

In order to study open quantum systems, that is, quantum physical systems in interaction with an environment, the *repeated interaction scheme* was first introduced in [2]. This setup has the

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advantage to furnish toy models for dissipative systems as quantum baths (see [4,5] for instance). Moreover, it corresponds to physical experiments as Haroche's ones on photons in a cavity as presented in [9,10].

The repeated interaction scheme was developed for the first time for classical systems in [7] using a mathematical framework based on deterministic dilatations of Markov chains and stochastic differential equations described in [1]. The main idea in the latter is that Markov processes are all obtained from deterministic dynamical systems on a product of two spaces when ignoring one of the two components. This mathematical approach is therefore relevant for a description of open classical systems in which we usually do not have access to all the information on the environment. However, even if the first motivation and the point of view taken in this article are physical, the repeated interaction setup, as described in [1], turns out to not be restricted only to physical dynamics since it more generally proposes an other way to understand Markovian evolutions.

These repeated interactions can be more precisely described as follows. We consider a system in interaction with a large environment. The latter is regarded as an infinite assembly of small identical pieces which act independently, one after the other, on the system during a small time step h. The state of each piece of the environment is randomly sampled from a probability measure representing a lack of knowledge on the environment which could arise from the inaccessibility of measurements of the environment or the impossibility to completely describe it. The advantage of this discrete-time model is that each interaction between the system and a piece of the environment is quite general, and can be explicitly described by a full Hamiltonian for instance, while the evolution of the system has a Markovian behavior facilitating the study of dynamics. A large class of Markov chains emerges in this model depending on the considered interaction and the probability measure on the environment.

In this article we focus on a usual physical interaction which is a system in a confining potential for which the Hamiltonian is of the form

$$H(p,q,P,Q) = \frac{\|p\|^2}{2} + V(q) + \frac{\|P\|^2}{2} + W(Q) + \eta(q)\beta(Q), \tag{1}$$

where q, p, Q and P are, respectively, the position and the momentum of the system and of each piece of the environment. In this operator, the map V represents the confining potential, the term $\|P\|^2/2 + W(Q)$ is the free dynamics of a part of the environment, that is, without interaction and $\eta(q)\beta(Q)$ is the coupling term. Such an interaction can be found in the literature and is similar to the one studied in [6] for instance.

In this model we are more particularly interested in the continuous-time limit of these repeated interactions, that is, the limit process given the evolution of the system when the interaction time h goes to 0. In the context of quantum systems, S. Attal and Y. Pautrat have shown in [2] that this model gives rise to usual quantum Langevin equations. For some classical systems, the limit evolution is proved in [7] to almost surely converge and in L^p to the solution of a stochastic differential equation. However, in [7], the systems are seen as dynamical systems in order to follow the description proposed by [1]. But this point of view requires some restrictions on the model; the state of each piece of the environment has to be sampled from a Gaussian measure and the limit stochastic differential equation has locally Lipschitz and linearly bounded coefficients corresponding to "linear" interactions between the system and the environment.

We here propose a different description of the repeated interaction scheme to free ourselves from these restrictions. Under some assumptions and after renormalizing the Hamiltonian H, we then prove the convergence in law of the Markov chain given the dynamics of the system to the