Available online at www.sciencedirect.com

ScienceDirect

Historia Mathematica 43 (2016) 65–86

HISTORIA MATHEMATICA

www.elsevier.com/locate/yhmat

In the footsteps of Julius König's paradox

Miriam Franchella

Dipartimento di Filosofia, Università degli Studi, Via Festa del Perdono 7, 20122 Milano, Italy

Available online 2 September 2015

Abstract

König's paradox, that he presented for the first time in 1905, preserved the same structure in all his papers: there was a number that at the same time was and was not finitely definable. Still, he changed the way for forming it, and both its consequences and its solutions changed as well. In the present paper we are going to follow the story of König's paradox, that is an intriguing mix of labelling, solving, criticising an "object" from different viewpoints and for different aims.

Zusammenfassung

Königs Paradoxon, das er zuerst im Jahr 1905 vorstellte, blieb in Hinblick auf seine Struktur in allen seinen Werken gleich: es handelt sich um eine Zahl, die gleichzeitig endlich definierbar und nicht endlich definierbar ist. Dennoch änderte er im Laufe der Zeit die Bildungsweise des Paradoxons, seine Konsequenzen und seine Auflösungen. Die vorliegende Arbeit folgt der Geschichte von Königs Paradoxon als einer faszinierenden Mischung von Etikettierungen, Auflösungen und Kritiken eines "Objektes" aus verschiedenen Blickwinkeln und mit unterschiedlichen Zielsetzungen.

MSC: 01A55; 03A05

Keywords: 20th century logic; Paradoxes; Set theory; Infinite

0. Introduction

Julius König presented what is known as his 'paradox' in 1905: there is a number that at the same time is and is not finitely definable. It was used by him as a step for (trying to) prove that the continuum was not well-ordered, and only later he shifted the focus to the contradiction itself and tried to find solutions. We are going to follow his efforts in our paper.

E-mail address: miriam.franchella@unimi.it.

http://dx.doi.org/10.1016/j.hm.2015.07.003

1. Biographical notes

Julius (Gyula) König was born in Györ on 16th December 1849.[1] He enrolled at the Medical Department of the University of Vienna and at the same time attended lectures on mathematics. After a short stay in Vienna, he went to Heidelberg, where he could devote himself both to natural sciences and to medicine. He got his doctoral degree in 1870 with a thesis on the theory of modular functions, although his first essay treated a medical subject (*Beiträge zur Theorie der elektrischen Nervenreizungen*). It seems that the arrival at Heidelberg of Leo Königsberger had been very influential on König's decision of choosing a mathematical subject for his doctoral work.[2] Then König spent half a year in Berlin, where he attended lectures of Weierstrass and Kronecker. He was appointed professor of the *Pedagógusképző Kara* (Teacher Training College) of Budapest in 1873 and Professor of the Third Department of the Technical University one year later. He was the Dean of the Faculty of engineering and also Rector of the University. During his life Julius König contributed to many areas of mathematics: algebra, analysis, geometry, number theory, set theory. His teaching activity extended outside the university as he held the view that the professors of the universities should always be in touch with secondary education, by guiding and helping work there. Hence, he established the programme of education in the field of commerce in Hungary and worked out the algebra part of the 1879 programme for secondary schools. He was one of the founders of the *Bolyai János Matematikai Társulat* (János Bolyai Mathematical Society) in Hungary and became a member of the *Magyar Tudományos Akadémia* (Hungarian Academy of Science) in 1889. In 1905 König retired (at the age of 56!), and this allowed him to spend the last part of his life working on his own approach to set theory, logic and arithmetic. He continued to give lectures at the university about these topics. A monograph about them was published in 1914, the year after his (sudden) death.

2. The framework of König's paradox

König's paradox was born and grew up inside set theory. Three things should be stressed.

a) In the first years of the 20th century no antinomy[3] about set theory had yet been discovered (see [Moore and Garciadiego, 1981]). Around 1895,[4] Cantor had noticed a strange effect that could come out of his set theory: the totality of all ordinal numbers could be well-ordered and, hence, if it were a set, an ordinal number could be assigned to it. Such an ordinal number would belong to the set (as the set is the totality of ordinal numbers) and would not belong to the set (according to a well-known theorem about well-ordered sets) at the same time. This means that the totality of ordinal numbers is an inconsistent multiplicity. According to Cantor, this gave us a precious piece of information about set theory: the totality of all ordinals cannot be a set. (Later, when a list of antinomies became fixed in the literature, this fact came to be read in the following terms: Cantor noticed the "antinomy of the maximum ordinal"

[1] See [Szénássy, 1992, 217–223, 233–255, 332–335] that contains also a complete bibliography of König.

[2] See http://www-history.mcs.st-andrews.ac.uk/Biographies/Konig_Julius.html.

[3] I recall here the Moore–Garciadiego distinction between a contradiction and an antinomy / a paradox: the latter is a contradiction that attracts the focus on itself, while the former is a means inside a proof by *reductio ad absurdum*. In the present paper I will then use indifferently the words 'antinomy' and 'paradox' in this sense.

[4] According to Bernstein [1905a, 187], Cantor had noticed the "contradiction" related to the existence of a maximum ordinal as early as 1895, and would have informed Hilbert in 1896. As Cantini [2009, 879] recalls, we have also the evidence of 1) a letter to Hilbert of 26.09.1897, where he sketches the argument that the totality of alephs cannot be at the same time well-defined and conceivable as a completed set; 2) a letter to Dedekind of 28.07.1899, where Cantor, on the basis of the theorem that ordinal numbers are linearly ordered, stated that the multiplicity of ordinal numbers is itself well ordered, but stressed that the it is not a set – hence no ordinal can be assigned to it and no contradiction can come out.