Studies in History and Philosophy of Modern Physics 51 (2015) 37–43

Contents lists available at ScienceDirect

Studies in History and Philosophy of Modern Physics

journal homepage: www.elsevier.com/locate/shpsb

Spacetime structure

Thomas William Barrett

Princeton University, Philosophy, 1879 Hall, Princeton, NJ 08544-1006, United States

ARTICLE INFO

Article history:
Received 13 January 2015
Received in revised form
11 June 2015
Accepted 26 June 2015
Available online 24 July 2015

Keywords:
Spacetime
Structure
Galilean spacetime
Newtonian spacetime
Minkowski spacetime

ABSTRACT

This paper makes an observation about the "amount of structure" that different classical and relativistic spacetimes posit. The observation substantiates a suggestion made by Earman (1989) and yields a cautionary remark concerning the scope and applicability of structural parsimony principles.

When citing this paper, please use the full journal title *Studies in History and Philosophy of Modern Physics*

1. Introduction

There is a story that is often told about the progression of classical spacetime theories.[1] We began long ago with Aristotelian spacetime. Aristotelian spacetime singles out a preferred worldline as the *center of the universe*. Then we moved to Newtonian spacetime and did away with this structure. Newtonian spacetime does not single out a preferred worldline, but it does single out a preferred inertial frame as the *rest frame*. Finally, we moved to Galilean spacetime and again did away with structure. Galilean spacetime does not even single out a preferred inertial frame.

This story provides a sense in which each of these classical spacetimes has "less structure" than its predecessors. It is natural to ask whether this progression towards less structure continues in the transition between classical and relativistic spacetimes. The purpose of this paper is to answer this question by investigating the structural relationships that hold between Galilean, Newtonian, and Minkowski spacetime. There is a precise sense in which Newtonian spacetime has more structure than both Galilean spacetime and Minkowski spacetime. But in this same sense, Galilean and Minkowski spacetime have *incomparable* amounts of structure; neither spacetime has less structure than the other. The progression towards a less structured spacetime therefore does not continue into the relativistic setting.

This discussion of spacetime structure will yield two modest philosophical payoffs. First, it will substantiate a remark made by Earman (1989). Earman has suggested, somewhat paradoxically, that Newtonian spacetime is a more natural stepping-stone to relativistic spacetimes than Galilean spacetime is. This discussion will provide one way of making Earman's suggestion perfectly precise. Second, this discussion will also yield a cautionary remark concerning the scope and applicability of the following methodological principle:

Structural parsimony: All other things equal, we should prefer theories that posit less structure.

This paper will provide an example of two physical theories that posit incomparable amounts of structure. In such cases, a structural parsimony principle is not applicable.

2. Structure preliminaries

We begin by explicating the idea of the "amount of structure" that a mathematical object has. We would like a clear and principled way to say when some mathematical object X has more or less structure than another mathematical object Y. One particularly natural way to compare amounts of structure appeals to the automorphisms, or symmetries, of a mathematical object.

E-mail address: thomaswb@princeton.edu

[1] See for example Geroch (1978), Earman (1989), and Maudlin (2012).

http://dx.doi.org/10.1016/j.shpsb.2015.06.004
1355-2198/

An automorphism of a mathematical object X is an invertible function from X to itself that preserves all of the structure of X. The automorphisms of an object bear a close relationship to the structure of the object. This relationship suggests the following kind of criterion for comparing amounts of structure:

SYM: A mathematical object X has more structure than a mathematical object Y if the automorphism group Aut(X) is "smaller than" the automorphism group Aut(Y).

The basic idea behind SYM is clear. If a mathematical object has more automorphisms, then it intuitively has less structure that these automorphisms are required to preserve. Conversely, if a mathematical object has fewer automorphisms, then it must be that the object has more structure that the automorphisms are required to preserve. The amount of structure that a mathematical object has is, in some sense, inversely proportional to the size of the object's automorphism group.[2]

The criterion SYM is intuitive, but it is imprecise. One way to make SYM precise is as follows:

*SYM**: A mathematical object X has more structure than a mathematical object Y if $\text{Aut}(X)\subsetneq\text{Aut}(Y)$.

The condition $\text{Aut}(X)\subsetneq\text{Aut}(Y)$ is one way to make precise the idea that Aut(X) is "smaller than" Aut(Y). SYM* makes intuitive verdicts in many simple cases of structural comparison. For example, one can verify that in general SYM* makes the following verdicts:

- A set X has less structure than a group $(X, \cdot)$.
- A set X has less structure than a topological space (X, τ).
- A vector space V has less structure than an inner product space (V, g).

The criterion SYM* is one particularly natural way to explicate the idea that an object X has "more structure" than another object Y. In what follows, we will use SYM* to compare the structure of Galilean, Newtonian, and Minkowski spacetime.[3] Earman has implicitly used SYM* to compare the structure of various classical spacetimes (Earman, 1989, Ch. 2). And indeed, SYM* makes the intuitive verdicts in all of these cases. Galilean spacetime has less structure than Newtonian spacetime, which in turn has less structure than Aristotelian spacetime. This paper simply extends Earman's discussion into the relativistic setting.

3. Spacetime preliminaries

Before applying SYM* to these spacetimes we need some preliminaries.[4] We first present the standard mathematical descriptions of Galilean, Newtonian, and Minkowski spacetime, and then discuss their automorphisms.

3.1. Spacetimes

Spacetime theories begin by specifying a smooth, connected, four-dimensional manifold M. Each point $p \in M$ represents the location of an "event" in spacetime. Galilean, Newtonian, and Minkowski spacetime all have the underlying manifold $M = \mathbb{R}^4$. They then endow $\mathbb{R}^4$ with different geometric structures.

Galilean spacetime is the tuple $(\mathbb{R}^4, t_{ab}, h^{ab}, \nabla)$. The smooth tensor fields t_{ab} and h^{ab} and the derivative operator ∇ are defined as follows:

$$t_{ab} = (d_a x^1)(d_b x^1)$$

$$h^{ab} = \left(\frac{\partial}{\partial x^2}\right)^a \left(\frac{\partial}{\partial x^2}\right)^b + \left(\frac{\partial}{\partial x^3}\right)^a \left(\frac{\partial}{\partial x^3}\right)^b + \left(\frac{\partial}{\partial x^4}\right)^a \left(\frac{\partial}{\partial x^4}\right)^b$$

∇ is the coordinate derivative operator on $\mathbb{R}^4$,

where $d_a x^i$ is the differential of the standard coordinate function $x^i : \mathbb{R}^4 \to \mathbb{R}$ and $(\partial/\partial x^i)^a$ is the standard ith coordinate vector field on $\mathbb{R}^4$. The coordinate derivative operator ∇ on $\mathbb{R}^4$ is defined to be the unique derivative operator that satisfies $\nabla_a (\partial/\partial x^i)^b = \mathbf{0}$ for each $i = 1, \ldots, 4$.[5] Importantly, we note that ∇ is flat, in the sense that its curvature field $R^a_{bcd} = \mathbf{0}$ everywhere on $\mathbb{R}^4$.

One interprets these geometric structures on Galilean spacetime as follows (Malament, 2012, Ch. 4.1). The field t_{ab} is a "temporal metric". It assigns a temporal length to vectors, and defines a preferred partitioning of Galilean spacetime into "simultaneity slices". The field h^{ab} is a "spatial metric". Given a vector ξ^a, one can use h^{ab} to (indirectly) assign a spatial length to it. Finally, the derivative operator ∇ endows $\mathbb{R}^4$ with a "standard of constancy". It specifies which trajectories through Galilean spacetime are geodesics.

Newtonian spacetime is obtained by adding a preferred notion of "rest" to Galilean spacetime. Specifically, it is the tuple $(\mathbb{R}^4, t_{ab}, h^{ab}, \nabla, \lambda^a)$, where t_{ab}, h^{ab} and ∇ are defined exactly as in Galilean spacetime, and

$$\lambda^a = \left(\frac{\partial}{\partial x^1}\right)^a.$$

The structures t_{ab}, h^{ab}, and ∇ are interpreted as above. The field λ^a singles out a preferred rest frame. It allows one to classify trajectories through Newtonian spacetime as "at rest" or "not at rest". A geodesic $\gamma : I \to \mathbb{R}^4$ with tangent field ξ^a is *at rest* if $\xi^a = c\lambda^a$ for some constant $c \in \mathbb{R}$.

It only remains to define Minkowski spacetime. *Minkowski spacetime* is the pair $(\mathbb{R}^4, \eta_{ab})$, with the Minkowski metric η_{ab} defined by

$$\eta_{ab} = (d_a x^1)(d_b x^1) - (d_a x^2)(d_b x^2) - (d_a x^3)(d_b x^3) - (d_a x^4)(d_b x^4).^6$$

The metric η_{ab} endows Minkowski spacetime with "lightcone structure". It allows one to classify a vector ξ^a at $p \in \mathbb{R}^4$ as *timelike* (if $\eta_{ab}\xi^a\xi^b > 0$) or *lightlike* (if $\eta_{ab}\xi^a\xi^b = 0$) or *spacelike* (if $\eta_{ab}\xi^a\xi^b < 0$). Timelike vectors at a point $p \in \mathbb{R}^4$ lie on the interior of the lightcone, lightlike vectors lie on the boundary of the lightcone, and spacelike vectors lie outside the lightcone.

[2] A criterion like SYM is used by North (2009) to compare the structure of Hamiltonian and Lagrangian mechanics and by Earman (1989) to compare various classical spacetime theories. See Halvorson (2011), Swanson & Halvorson (2012), Curiel (2014), and Barrett (2015) for related discussions of structure and classical mechanics.

[3] SYM* has some shortcomings. In particular, it is overly sensitive to the set that underlies a mathematical object. There is a sense in which a topological space (X, τ) has more structure than a set Y even when the sets X and Y are distinct. It is not the case, however, that $\text{Aut}(X, \tau) \subset \text{Aut}(Y)$ (since functions from X to itself are different from functions from Y to itself), so SYM* is incapable of capturing this sense. But this will not be problematic for our purposes; all of the spacetimes that we consider have the same underlying set $\mathbb{R}^4$.

[4] The reader is encouraged to consult Malament (2012) or Wald (1984) for details.

[5] For proof that the coordinate derivative operator is unique see Malament (2012, Prop. 1.7.11). One can easily verify that Galilean spacetime, so defined, is a classical spacetime in the sense of Malament (2012, p. 249).

[6] Note that $\nabla_a \eta_{bc} = \mathbf{0}$, where ∇ is the derivative operator defined above. So the coordinate derivative operator ∇ on $\mathbb{R}^4$ is the unique derivative operator compatible with the Minkowski metric.