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ORIGINAL ARTICLE

# Comparison of optimal homotopy asymptotic method and homotopy perturbation method for strongly non-linear equation

**Fazle Mabood** *

*Department of Mathematics, Edwardes College Peshawar, KPK 25000, Pakistan*

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**Abstract** In this paper, we employ an approximate analytical method, namely the optimal homotopy asymptotic method (OHAM), to investigate a thin film flow of a third grade fluid down an inclined plane and provided accurate solution unlike other erroneous results available in the literature. The variation of the velocity field for different parameters is compared with the numerical values obtained by the *Runge–Kutta Fehlberg fourth–fifth order* numerical method and with the homotopy perturbation method (HPM). Finally, it was found that for all values of parameters OHAM agrees well with the numerical disparate HPM.


## 1. Introduction

Most scientific phenomena are inherently nonlinear such as heat transfer, and many of them have no analytical solution. Therefore, many different methods have been established by researchers to overcome such nonlinear problems. These methods include the artificial parameter method by (He, 2006a,b), the variational iteration method by (He, 2000), the homotopy analysis method by (Liao, 2003), the homotopy perturbation method by (He, 2006c) and the optimal homotopy asymptotic method by (Marinca and Herisanu, 2008) among others. The homotopy perturbation method (HPM) provides an approximate analytical solution in a series form. HPM has been widely used by numerous researchers successfully for different physical systems such as, bifurcation, asymptotology, nonlinear wave equations, oscillators with discontinuities by (He, 2004ab, 2005a,b), reaction-duffision equation and heat radiation equation by (Ganji and Rajabi, 2006; Ganji and Sadighi, 2006) and MHD Jeffery–Hamel problem by (Moghimi et al., 2011).

Significant classes of fluids commonly used in industries are non-Newtonian fluids. The applications of these fluids arise in areas such as synthetic fibers, food stuffs, drilling oil and gas wells, extrusion of molten plastics and polymers among others. The related literature indicates that the third grade fluid has been investigated by many researchers for different geometries and with different techniques.

Here, we consider the steady uni-directional flow of an incompressible third-grade fluid down a uniform inclined plane. For the third grade fluid, the first four terms of Taylor

* Tel.: +92 3339142794.
E-mail address: mabood1971@yahoo.com.
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series are using the stress rate of strain relation. The third grade fluid models are complicated due to a large number of physical parameters that have to be determined experimentally.

The steady flow of third grade fluid in a bounded domain with Dirichlet boundary conditions analyzed by Adriana et al., 2008. Bresch and Lemoine (1999) have shown the existence of the solutions for non-stationary third-grade fluids and used homogenous boundary condition for the global and local existence of the fluid velocity equation. Many researchers (Zhang and Li, 2005; Busuioc et al., 2008; Khan and Mahmood, 2012; Siddiqui et al., 2008; Hayat et al., 2008, 2009; Kumaran et al., 2012) have investigated thin film flow of the third grade fluid, in addition Hameed and Ellahi, 2011 studied thin film flow for MHD fluid on moving belt. Moreover, Elahi and Riaz, 2010; Ellahi et al., 2011; Ellahi, 2012 successfully provided the series solution for non-Newtonian MHD flow with variable viscosity in a third grade fluid and discussed heat transfer in porous cylinder.

The optimal homotopy asymptotic method is an approximate analytical tool that is simple and straightforward and does not require the existence of any small or large parameter as does the traditional perturbation method. The optimal homotopy asymptotic method (OHAM) has been successfully applied to a number of nonlinear problems arising in fluid mechanics and heat transfer by various researchers (Herisanu et al., 2008; Mabood et al., 2013a,b; Marinca and Herisanu, 2008, 2010a,b).

Mathematical modeling of non-Newtonian fluid flow gives rise to nonlinear differential equations. Many numerical and analytical techniques have been proposed by various researchers. An efficient approximate analytical solution will find enormous applications. In this paper, we have solved the governing nonlinear differential equation of the present problem using OHAM and compared with numerical and HPM. It is important to mention here that the approximate analytical and numerical solutions are in a good agreement but better than the results of Siddiqui et al., 2008.

This paper is organized as follows: First in Section 2, governing equations of the problem are presented. In Section 3 we described the basic principles of OHAM. The OHAM solution is given in Section 4. In Section 5, outlines of HPM are discussed with HPM solution. In Section 6, we analyzed the comparison of the solution using OHAM with the numerical method and existing solution of HPM. Section 7 is devoted for the concluding remarks.

## 2. Governing equation

The thin film flow of an incompressible third grade fluid down on an inclined plane with inclination $\alpha \neq 0$ is governed by the following nonlinear boundary value problem in a dimensionless form (Siddiqui et al., 2008).

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + m = 0 \tag{1}$$

Subject to the boundary conditions:

$$u(0) = 0, \qquad \frac{du}{dy} = 0 \quad \text{at} \quad y = 1 \tag{2}$$

As $m = \dfrac{g\rho \sin\alpha}{\mu}$, $\beta = \dfrac{(\beta_2 + \beta_3)}{\mu}$

where $u$ is the fluid velocity, $\rho$ is the density, $\mu$ is the dynamic viscosity, $\beta_2$ and $\beta_3$ are the material constants of the third grade fluid, $g$ is acceleration due to gravity.

## 3. Basic principles of OHAM

We review the basic principles of OHAM as expounded in Herisanu et al., 2008 and other researchers (Mabood et al., 2013a; Marinca and Herisanu, 2008).

(i) Consider the following differential equation:

$$A[v(x)] + a(x) = 0, \quad x \in \Omega \tag{3}$$

where $\Omega$ is problem domain, $A(v) = L(v) + N(v)$, where $L$, $N$ are linear and nonlinear operator, $v(x)$ is an unknown function, $a(x)$ is a known function,

(ii) Construct an optimal homotopy equation as:

$$(1-p)[L(\phi(x;p) + a(x)] - H(p)[A(\phi(x;p) + a(x)] = 0 \tag{4}$$

where $0 \leqslant p \leqslant 1$ is an embedding parameter, $H(p) = \sum_{k=1}^{m} p^k C_k$ is auxiliary function on which the convergence of the solution is greatly dependent. The auxiliary function $H(p)$ also adjusts the convergence domain and controls the convergence region.

(iii) Expand $\phi(x; p, C_j)$ in Taylor's series about $p$, one has an approximate solution:

$$\phi(x; p, C_j) = v_0(x) + \sum_{k=1}^{\infty} v_k(x, C_j)p^k, \quad j = 1, 2, 3, \ldots \tag{5}$$

Many researchers have observed that the convergence of the series Eq. (5) depends upon $C_j$, $(j = 1, 2, \ldots, m)$, if it is convergent then, we obtain:

$$\tilde{v} = v_0(x) + \sum_{k=1}^{m} v_k(x; C_j) \tag{6}$$

(iv) Substituting Eq. (6) in Eq. (4), we have the following residual:

$$R(x; C_j) = L(\tilde{v}(x; C_j)) + a(x) + N(\tilde{v}(x; C_j)) \tag{7}$$

If $R(x; C_j) = 0$, then $\tilde{v}$ will be the exact solution. For nonlinear problems, generally this will not be the case. For determining $C_j$, $(j = 1, 2, \ldots, m)$, Galerkin's Method, Ritz Method or the method of least squares can be used.

(v) Finally, substitute these constants in Eq. (7) and one can get the approximate solution.

## 4. Solution of the problem via OHAM

According to the OHAM, applying Eq. (4) to Eq. (1):

$$(1-p)(u'' + m) - H(p, C_i)\{u'' + 6\beta u'^2 u'' + m\} = 0 \tag{8}$$

where primes denote differentiation with respect to $y$.