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ORIGINAL ARTICLE

A new decomposition technique for solving a system of linear equations

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Abstract In this paper, we use a new decomposition technique to suggest and consider some new iterative methods for solving system of linear equations. We prove that these iterative methods are similar to the iterative methods derived by using homotopy perturbation method and Adomian decomposition method. We consider the elliptic partial differential equation along with other several numerical examples to illustrate the efficiency and performance of our results. Our results can be viewed as an improvement and extensions of the previously known results.

1. Introduction

In recent years, several methods and techniques have been developed to solve system of linear equations. Liu (2011), Keramati (2009), Noor (2010) and Noor et al. (2013a) have used homotopy perturbation method to derive iterative methods for solving linear (nonlinear) equations. Noor et al. (2013b) have used Adomian decomposition method to develop iterative methods for system of linear equations. Babolian et al. (2004) have used Adomian decomposition method to derive an iterative method similar to the Jacobi iterative method for solving system of linear equations. Allahviranloo (2005) used Adomian decomposition method for fuzzy system of linear equations. There are many publication in the field of analytical surveys using homotopy perturbation methods and other techniques, see, for example, Ganji (2006, 2012), Ganji and Sadighi (2006), Jalal and Ganji (2010, 2011) and Jalal et al. (2010, 2012). In the implementation of the Adomian (1989, 1994) decomposition method, one has to calculate the derivatives of the so-called Adomian polynomials, which is itself a difficult problem. To overcome this drawback, we use a different type of decomposition which is essentially due to Daftardar-Gejji and Jafari (2006), to develop the iterative methods for solving the system of linear equations. Noor (2006,2007), Noor and Noor (2006a,b), Noor et al. (2006c) and Noor et al. (2010a,b) have used the same decomposition technique for solving nonlinear equations. This decomposition method does not involve the high-order differentials of the function and is very simple as compared with Adomian decomposition technique. In this paper, we use this new decomposition method to develop iterative methods for solving system of linear equations. We show that our results obtained by using new decomposition technique are the same as derived by Liu (2011) and

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Noor et al. (2013b) by using the homotopy perturbation method and Adomian decomposition method, respectively. This is the main motivation of this paper. By using new iterative methods we solve the elliptic partial differential equation and it is well known that the elliptic partial differential equations have applications in almost all areas of mathematics and frequent applications in engineering and physics. We also give several numerical examples to demonstrate the efficiency and performance of our results.

2. Iterative methods

Consider the system of linear equations

$$AX = b, \tag{2.1}$$

where

$A = [a_{ij}], \quad X = [x_j] \quad$ and $b = [b_j], \quad i = 1, 2, \cdots, n,$
$j = 1, 2, \cdots, n.$

It is well-known that systems of linear Eq. (2.1) arise in studies in many areas such as engineering, industrial science and so on. For example, in digital image and signal processing, especially in compressed sensing, biomedical engineering, systems and control science, machine learning and so on. For the formulation and applications of the system of linear Eq. (2.1), see Burden and Faires (2001) and the references therein. We decompose the system of linear Eq. (2.1) in such way, which is useful in developing the iterative methods. For an auxiliary parameter $\hbar \neq 0$, any splitting matrix Q and an auxiliary matrix H, we can decompose the system of linear Eq. (2.1) as follows:

$$QX + (\hbar HA - Q)X = \hbar Hb. \tag{2.2}$$

Let W_0 be the initial approximation of X, then, Eq. (2.2) can be written as:

$$QX = W_0 + [(Q - \hbar HA)X + \hbar Hb - W_0] \tag{2.3}$$

Eq. (2.3) can be written as:

$$L(X) = C + M(X), \tag{2.4}$$

where

$$L(X) = QX, \tag{2.5}$$

$$C = W_0, \tag{2.6}$$

$$M(X) = [(Q - \hbar HA)X + \hbar Hb - W_0] \tag{2.7}$$

Here we use a new decomposition technique, which is mainly due to Daftardar-Gejji and Jafari (2006), to construct a family of iterative methods. In this technique, the decomposition of the operator $M(X)$ is quite different than that of Adomian (1989, 1994) decomposition. See also He (1999), Babolian et al. (2004) and Yusufoglu (2009) for other techniques.

The main idea of this technique is to look for a solution of Eq. (2.4) having the series form

$$X = \sum_{i=0}^{\infty} X_i. \tag{2.8}$$

The operator M is decomposed (Daftardar-Gejji and Jafari (2006)) as

$$M(X) = M(X_0) + \sum_{i=1}^{\infty}\left\{ M\left(\sum_{j=0}^{i} X_j\right) - M\left(\sum_{j=0}^{i-1} X_j\right)\right\} \tag{2.9}$$

Combining (2.4), (2.8), and (2.9), we have

$$L\left(\sum_{i=0}^{\infty} X_i\right) = C + M(X_0) + \sum_{i=1}^{\infty}\left\{ M\left(\sum_{j=0}^{i} X_j\right) - M\left(\sum_{j=0}^{i-1} X_j\right)\right\} \tag{2.10}$$

By using (2.5) and (2.10), we have

$$Q\left(\sum_{i=0}^{\infty} X_i\right) = C + M(X_0) + \sum_{i=1}^{\infty}\left\{ M\left(\sum_{j=0}^{i} X_j\right) - M\left(\sum_{j=0}^{i-1} X_j\right)\right\} \tag{2.11}$$

Thus, we have the following iterative scheme:

$$\begin{gathered} Q(X_0) = C, \\ Q(X_1) = M(X_0), \\ Q(X_2) = M(X_0 + X_1) - M(X_0), \\ \vdots \\ Q(X_{m+1}) = M\left(\sum_{j=0}^{m} X_j\right) - M\left(\sum_{j=0}^{m-1} X_j\right). \quad m = 1, 2, \cdots. \end{gathered} \tag{2.12}$$

From (2.7) and (2.12), we have

$$\begin{aligned} &Q(X_0) = C, \\ &Q(X_1) = (Q - \hbar HA)X_0 + \hbar Hb - W_0, \\ &Q(X_2) = (Q - \hbar HA)X_1, \\ &\vdots \\ &Q(X_{m+1}) = (Q - \hbar HA)X_m, \quad m = 1, 2, \cdots. \end{aligned} \tag{2.13}$$

From (2.13), we get

$$\begin{cases} X_0 = Q^{-1}W_0, \\ X_1 = (I - \hbar Q^{-1}HA)X_0 + Q^{-1}(\hbar Hb - W_0), \\ X_m = (I - \hbar Q^{-1}HA)X_{m-1}, \quad m = 2, 3, 4, \cdots. \end{cases} \tag{2.14}$$

Taking initial approximation $W_0 = \hbar Hb$, we have

$$\begin{cases} X_0 = \hbar(Q^{-1}H)b, \\ X_m = (I - \hbar Q^{-1}HA)^m \hbar(Q^{-1}H)b, \quad m = 1, 2, 3, \cdots. \end{cases} \tag{2.15}$$

Thus, from (2.8) and (2.15), we have the series solution

$$X = \sum_{k=0}^{\infty} X_k = \sum_{k=0}^{\infty} (I - \hbar Q^{-1}HA)^k \hbar(Q^{-1}H)b. \tag{2.16}$$

Formula (2.16) gives exactly the same series solution obtained by using homotopy perturbation technique in Liu (2011) and Adomian decomposition method Noor et al. (2013b). However, our technique of the derivation of the series solution is quite easy and natural one. This technique does not involve the computation of the Adomian polynomials, which is itself a difficult problem. For the convergence analysis of series (2.16), see Liu (2011). The series (2.16) converges if and only if $\rho(I - \hbar Q^{-1}HA) < 1$, see Liu (2011). The auxiliary parameter $\hbar \neq 0$, and the auxiliary matrix H are chosen properly so that the series (2.16) converges.