Computational Materials Science 83 (2014) 57–63

Contents lists available at ScienceDirect

Computational Materials Science

journal homepage: www.elsevier.com/locate/commatsci

Maximizing the effective stiffness of laminate composite materials

Zhi Hao Zuo, Yi Min Xie *

Centre for Innovative Structures and Materials, School of Civil, Environmental and Chemical Engineering, RMIT University, GPO Box 2476, Melbourne 3001, Australia

ARTICLE INFO

Article history:
Received 9 August 2013
Received in revised form 8 October 2013
Accepted 4 November 2013
Available online 23 November 2013

Keywords:
Effective stiffness
Laminate composite
Design optimization
Poisson's ratio
Rule of mixture

ABSTRACT

The effective stiffness of laminate composites can be expressed explicitly and accurately as a function of several variables such as volume fractions and elastic constants of the constituent phases. Based on the stiffness function, an optimization procedure is proposed in this paper to maximize the effective Young's moduli of laminate composites in both longitudinal and transverse directions with respect to a number of design variables. By solving such a constrained optimization problem, a laminate composite can be designed by finding the optimal material properties of the constituent phases and their volume fractions. The effects of the volume fractions, the Young's moduli and the Poisson's ratios of the constituent phases in the effective composite stiffness are demonstrated through various design cases. It is shown that the optimized effective Young's moduli can reach values much higher than the well-known approximated Voigt estimation. Dramatic increases in the effective stiffness have also been found when the Poisson's ratios of the constituent phases approach the thermodynamic limits of −1 and 0.5. It is envisaged that with the proposed approach and modern fabrication technologies, laminate composites with exceptional effective stiffness can be easily designed and manufactured.

1. Introduction

The elastic properties of composites have been extensively studied. Various simplified estimations have been proposed on the elastic matrices of composites, among which the Voigt estimation [1] and the Reuss estimation [2] have been widely adopted and proven able to provide the upper and lower bounds respectively for the effective bulk and shear moduli [3].

Most isotropic materials have a Poisson's ratio within a very narrow range between 1/4 and 1/3 [4]. By neglecting the Poisson's effect, an approximated Voigt estimation of the effective stiffness of a composite can be obtained as the weighted volume average of those of the isotropic constituent phases [3]. Nevertheless, recent works indicate that this approximated bound can actually be exceeded [5–7] if the constituent phases have exceptional Poisson's ratios, such as negative Poisson's ratios [8,9] with which the material is called auxetic or dilational due to the fact that it will expand transversally when pulled in one direction. Liu et al. [5] showed that the effective stiffness of composites can increase beyond the approximated Voigt estimation and the Young's moduli of the constituent phases, if one of the phases is nearly incompressible, i.e. the Poisson's ratio approaching 0.5. Kocer et al. [6] and Lim [7] discovered that the effective stiffness of a two-phase laminate composite can exceed the approximated Voigt estimation if one phase has positive Poisson's ratio and the other phase negative Poisson's ratio.

The stiffness is one of the most sought-after material properties that have practical applications in engineering. Composite materials provide such an opportunity to reach exceptional stiffness by combining the constituent phases of distinct material properties. The elastic properties of composites are affected not only by the Poisson's ratios of the constituent phases, but also by other factors such as the Young's moduli of the constituent phases, the volume proportion of each phase, and the topology of the material micro structure, i.e. how the phases are geometrically combined to form the composite. Based on the conventional layered composite form of laminate with two phases, Liu et al. [5] and Lim [7] independently calculated the composite effective stiffness explicitly with the variables of the elastic properties of the isotropic laminas and their volume fractions. In the mathematical point of view, this forms a basis for calculating the theoretical maximum value of the effective stiffness of this type of composites.

This paper focuses on two-phase laminates and investigates the maximum effective stiffness of layered composites. Various cases are studied to calculate the maximum effective stiffness under different conditions such as prescribing or not prescribing the elastic properties of the constituent phases. Exceptional effective stiffness will be obtained even if the Poisson's ratios of the constituent phases are not near the limits of −1 and 0.5, by mathematically finding the suitable constituent phases and assigning a proper volume proportion.

* Corresponding author. Tel.: +61 3 99253655; fax: +61 3 96390138.
E-mail address: mike.xie@rmit.edu.au (Y.M. Xie).

0927-0256/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.commatsci.2013.11.010

2. Calculation of effective stiffness

The laminate composite with isotropic constituent phases is a transversally isotropic material, i.e. the material properties are isotropic in the layered plane. Therefore there are two independent effective Young's moduli in two directions respectively: the longitudinal stiffness and the transverse stiffness. Two corresponding loading conditions for testing these two effective Young's moduli are shown in Fig. 1.

The Hooke's law in the compliance form for the laminate composite indicates the constitutive equations and is expressed in the following compact form.

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & Symm & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \mathbf{S}\boldsymbol{\sigma} \tag{1}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress vector respectively, $\mathbf{S}$ is the compliance matrix. Due to the simple loading conditions, the effective stiffness can be easily obtained from the inverse of corresponding item in the compliance matrix. For instance, the effective transverse stiffness can be obtained by $1/S_{33}$, namely $\sigma_{zz}/\varepsilon_{zz}$ since σ_{xx} and σ_{yy} are both zero under the loading in Fig. 1(b). By solving a system of equations involving the constitutive equations (Eq. (1)), static equilibria and kinematics [5], the effective Young's moduli can be explicitly calculated based on the Young's moduli, Poisson's ratios and the volume fractions of the two phases. Since the derivation is not the concern of this paper and can be referred to Liu et al. [5], the explicit formulae are given directly in the following.

$$\begin{aligned} E_x^{eff} &= \frac{1}{S_{11}} = E_x^{eff}(E_A, E_B, \nu_A, \nu_B, \phi_A, \phi_B) \\ &= (\phi_A E_A + \phi_B E_B) + \frac{\phi_A \phi_B E_A E_B (\nu_A - \nu_B)^2}{\phi_A E_A (1 - \nu_B^2) + \phi_B E_B (1 - \nu_A^2)} \end{aligned} \tag{2}$$

$$\begin{aligned} E_z^{eff} &= \frac{1}{S_{33}} = E_z^{eff}(E_A, E_B, \nu_A, \nu_B, \phi_A, \phi_B) \\ &= \frac{E_A E_B}{\phi_A E_B + \phi_B E_A - \frac{2\phi_A \phi_B (\nu_A E_B - \nu_B E_A)^2}{(1-\nu_A)\phi_B E_B + (1-\nu_B)\phi_A E_A}} \end{aligned} \tag{3}$$

where E_x^{eff} and E_z^{eff} are the longitudinal and transverse effective Young's moduli, E_A and E_B are the Young's moduli of the two isotropic phases A and B, ν_A and ν_B are the Poisson's ratios, and Φ_A and Φ_B are the volume fractions respectively.

It is seen that both effective Young's moduli are functions of six unknowns. However, the volume fractions of the two phases are not independent – they add up to 100%, i.e.

$$\phi_B = 1 - \phi_A \tag{4}$$

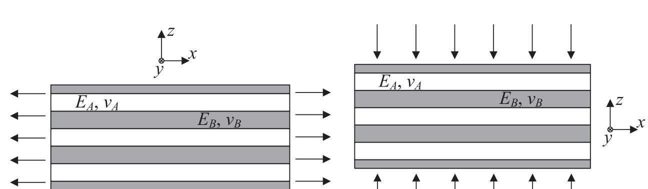

Fig. 1. The loading conditions for the two effective Young's moduli: (a) effective longitudinal stiffness; (b) effective transverse stiffness.

Furthermore, it is obvious that the effective stiffness of the composite will increase accordingly if we increase the Young's modulus of either phase. Therefore, it is beneficial to define a normalized effective stiffness by setting the Young's modulus E_A as a constant reference. As a result, E_A will drop out from the formulae of the effective stiffness. Herein, we define the normalized effective Young's moduli in the longitudinal and transverse directions as follows.

$$\overline{E_x^{eff}} = \frac{E_x^{eff}}{E_A} \tag{5}$$

$$\overline{E_z^{eff}} = \frac{E_z^{eff}}{E_A} \tag{6}$$

In the meantime, extra factors α and β are introduced to replace the Young's modulus E_B for a clearer form of the formulae which is shown later.

$$\alpha = \frac{1}{\beta} = \frac{E_B}{E_A} \tag{7}$$

By substituting Eqs. (4)–(7) into Eqs. (2) and (3) respectively, we finally obtain the normalized effective Young's moduli as functions of four unknowns in the following.

$$\begin{aligned} \overline{E_x^{eff}}(\phi_A, \nu_A, \nu_B, \alpha) &= \alpha + \phi_A - \alpha\phi_A \\ &\quad + \frac{\alpha\phi_A(1-\phi_A)(\nu_A - \nu_B)^2}{\phi_A(1-\nu_B^2) + \alpha(1-\phi_A)(1-\nu_A^2)} \end{aligned} \tag{8}$$

$$\overline{E_z^{eff}}(\phi_A, \nu_A, \nu_B, \beta) = \frac{1}{\phi_A + (1-\phi_A)\beta - \frac{2\phi_A(1-\phi_A)(\nu_A - \beta\nu_B)^2}{(1-\nu_A)(1-\phi_A) + (1-\nu_B)\phi_A\beta}} \tag{9}$$

3. Optimization for maximum effective stiffness

3.1. Problem statement

It is known from the strain energy formulation that the Poisson's ratios ν_A and ν_B of the two isotropic phases range within the domain $(-1, 0.5)$. Besides, it is obvious that the volume fraction Φ_A falls in the domain $[0, 1]$. In an optimization problem formulation, the above unknowns are taken as the design variables to search for the objective function, which in this case, is either of the normalized effective stiffness. The feasible domains of Φ_A, ν_A and ν_B are then implemented as constraints. On the other hand, it is envisaged that the effective Young's moduli are monotonically increasing functions of E_B, and thus are monotonic with α for $\overline{E_x^{eff}}$ and β for $\overline{E_z^{eff}}$. Therefore it is practical to treat α and β as prescribed constants, leaving only three unknowns for the effective stiffness functions: Φ_A, ν_A and ν_B. Then the maximization problem of the normalized effective longitudinal stiffness can be formulated as follows.

$$\text{Find } \{\phi_A, \nu_A, \nu_B\}, \text{ maximize } \overline{E_x^{eff}} \text{ which is defined in Eq. (8)} \tag{10}$$

subject to

$$0 \le \phi_A \le 1 \tag{11}$$

$$-1 < \nu_A < 0.5 \tag{12}$$

$$-1 < \nu_B < 0.5 \tag{13}$$

On the other hand, the complex form of the formula for the normalized effective transverse stiffness brings unnecessary difficulties for differentiating that is needed to solve for the function