Applied Mathematical Modelling 40 (2016) 984–992

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Hamiltonian system-based analytic modeling of the free rectangular thin plates' free vibration

Rui Li [a,b,*], Bo Wang [a], Gang Li [a], Bin Tian [c]

[a] State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China
[b] State Key Laboratory of Digital Manufacturing Equipment & Technology, Huazhong University of Science and Technology, Wuhan 430074, PR China
[c] Department of Science and Technology, China Road and Bridge Corporation, Beijing 100011, PR China

ARTICLE INFO

Article history:
Received 25 February 2015
Revised 3 May 2015
Accepted 18 June 2015
Available online 16 July 2015

Keywords:
Rectangular thin plate
Free vibration
Hamiltonian system
Symplectic superposition method

ABSTRACT

Analytic free vibration solutions of free rectangular thin plates with or without an elastic foundation are obtained by using an up-to-date Hamiltonian system-based symplectic superposition method. Such boundary value problems are known to be very difficult and they were generally solved by the approximate/numerical methods. The present analytic solutions are expected to be the benchmarks for future verification of other methods. The advantage of the method is that the solution procedure is conducted in the symplectic space, where the symplectic eigen expansion is valid and the predetermination of the solution form is avoided. This significantly extends the approach to the analytic solutions of similar problems.

1. Introduction

Free vibration of rectangular thin plates is of fundamental importance in both mechanical and civil engineering. The issues have been investigated for many years, and great achievements have been gained. The classical solutions by different methods such as the Rayleigh–Ritz method, Galerkin method, and series method were systematically introduced in the famous technical report by Leissa [1], where a comprehensive set of results for the frequencies and mode shapes of free vibration of plates were archived and they are very helpful to the researchers in the field.

In recent years, more approximate/numerical methods have been developed to solve the plate vibration problems. Representative methods include the differential quadrature method [2,3], discrete singular convolution method [4,5], half boundary method [6], extended Kantorovich method [7], meshless method [8], symplectic semi-analytic method [9], etc.

Although there are abundant achievements, the free vibration of plates is still the subject of a continuous interest in the solid mechanics community. One of the key issues is that the accurate analytic solutions are far beyond completeness although there are a variety of approximate/numerical solutions with different levels of precision. Actually, by the traditional semi-inverse method, the analytic free vibration solutions of rectangular thin plates were restricted to those with two opposite edges simply supported; and it is rather difficult to find the semi-inverse solutions of the other cases which satisfy both the governing equation and the boundary conditions in view of the mathematical complexity of those boundary value problems. To address this, some researchers have made great efforts and much progress has been made. Gorman proposed a semi-inverse superposition method

* Corresponding author at: State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China. Tel.: +86 15840912019.
E-mail address: ruili@dlut.edu.cn, ruili19851985@yahoo.com (R. Li).

http://dx.doi.org/10.1016/j.apm.2015.06.019
S0307-904X(15)00402-3/

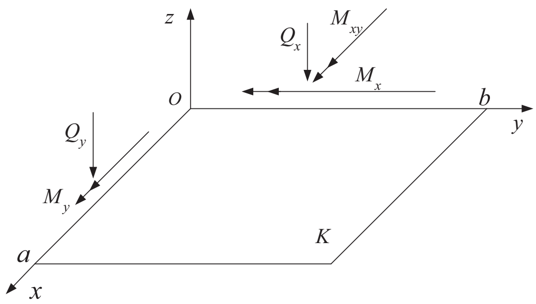

Fig. 1. Coordinates and positive internal forces of a rectangular foundation plate with foundation modulus *K*.

based on the use of Lévy-type single series to obtain the free vibration solutions of rectangular thin plates [10]. Zhang et al. [11] used the series expansion to analyze the rectangular plates with arbitrary elastic edge restraints. Zhong et al. [12] applied the finite integral transform method [13] to the free vibration of foundation plates. Lim et al. [14] developed the analytic solutions for the free vibration of Lévy-type rectangular thin plates by the symplectic elasticity approach. The symplectic elasticity approach was originated by Zhong's group in the 1990s [15], and was developed to form a systematic methodology in 2002 when the first monograph on it was published in Chinese [16]. In 2009, the English edition of the monograph was published, named *Symplectic Elasticity* [17], which has attracted much attention from different research areas. Actually, the symplectic methodology has shown the broad applicability [18] and has been applied in elasticity [16,17,19,20], symplectic numerical methods [21–23], fracture mechanics [24,25], piezoelectricity [26], functionally graded effects [27], magneto-electro-elasticity [28], soft matter [29], etc.

Hu et al. [30] recently further developed the method to obtain the vibration solutions of rectangular orthotropic plates with combinations of clamped and simply supported edges. Similar treatment was adopted by Xing and Xu [31] to solve the same problems. It should be noted that the conventional symplectic method has not exhibited the ability to solve the vibration of the plates with free edges while without two simply supported opposite edges.

The present study is the first endeavor to extend the up-to-date symplectic superposition method [19,32] to the free vibration of free rectangular thin plates. The paper focuses on the free plates with or without an elastic foundation. The analytic solutions are obtained in a rigorous step-by-step manner, with no assumptions of solution forms made. Rationality of the solution procedure and accuracy of the results make the present method a promising one which allows for the exploration of new analytic solutions that are difficult to obtain by other analytic methods.

2. Hamiltonian system-based governing equation

The generalized variational principle with two kinds of variables for the free vibration problem of a thin Winkler-type foundation plate is

$$\delta\Pi_2 = 0 \tag{1}$$

where the generalized potential energy functional

$$\Pi_2 = \iint_\Omega \left\{ -M_x \frac{\partial^2 w}{\partial x^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} - M_y \frac{\partial^2 w}{\partial y^2} - \frac{1}{2D(1-\nu^2)}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2\right] - \frac{\rho h\omega^2 - K}{2}w^2 \right\} \mathrm{d}x\mathrm{d}y \tag{2}$$

is defined over the plate domain Ω in the coordinate system (x, y) $(0 \le x \le a$, and $0 \le y \le b)$. w is the flexural displacement of the vibrating plate, M_x and M_y are the bending moments, M_{xy} is the torsional moment, ν is the Poisson's ratio, D is the flexural rigidity, ρ is the mass density, h is the plate thickness, ω is the circular frequency, and K is the foundation modulus. The coordinates and positive internal forces of such a plate are shown in Fig. 1. Assuming the independence of M_x, M_y, M_{xy} and w and the arbitrariness of their variations, Eq. (1) yields the basic equations of the plate, two of which are $M_x = -D(\partial^2 w/\partial x^2 + \nu\partial^2 w/\partial y^2)$ and $M_{xy} = -D(1-\nu)\partial^2 w/\partial x\partial y$. Substituting them into Eq. (2) and introducing $\theta = \partial w/\partial y$ and the Lagrange multiplier T, a new functional Π_H is constructed:

$$\Pi_H = \iint_\Omega \left[\frac{D}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{D}{2}\left(\frac{\partial \theta}{\partial y}\right)^2 + D\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial \theta}{\partial y} + D(1-\nu)\left(\frac{\partial \theta}{\partial x}\right)^2 - \frac{D}{2(1-\nu^2)}\left(\frac{M_y}{D} + \frac{\partial \theta}{\partial y} + \nu\frac{\partial^2 w}{\partial x^2}\right)^2 + T\left(\theta - \frac{\partial w}{\partial y}\right) - \frac{\rho h\omega^2 - K}{2}w^2 \right] \mathrm{d}x\mathrm{d}y. \tag{3}$$