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Numerical method for stability analysis of functionally graded beams on elastic foundation

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ABSTRACT

This paper considers the problems of stability of non-uniform and axially functionally graded (FG) Euler–Bernoulli beams on elastic foundation. The boundary conditions of the beam are given in general form covering different classical supporting conditions. Furthermore, the boundary conditions are transformed into convenient form in order to avoid handling infinity values of the stiffness coefficients. The method of initial parameters in differential form is used for the numerical solution of the problem. The solution of the posed boundary value problem is reduced to the iterative solution of two initial value problems and homogeneous linear algebraic system of order two. The Richardson extrapolation method is employed to improve the accuracy of results. The performance of the method proposed has been validated by several numerical examples.

1. Introduction

The buckling problems of non-uniform and/or axially functionally graded (FG) beams/columns have been overviewed in monographs [1–4]. However, exact analytical solutions exist only for certain particular cases of the distributions of loads and flexural rigidity [3–7].

In more general cases of the distributions of flexural rigidity, numerical methods should be employed. In the well-known paper of Eisenberger [8], a number of examples are given for buckling of non-uniform columns with different distributions of rigidity and loads. The highly accurate results obtained in [8] in terms of series expansion are used often for comparison and evaluation of different numerical methods.

The numerical solutions of buckling problems of non-uniform and/or axially FG beams/columns under concentrated compressive load have been recently considered in number of papers [9–23]. In the paper of Sapountzakis and Tsiatas [9] the boundary element method is developed for the elastic flexural buckling analysis of composite Euler–Bernoulli beams of arbitrary variable cross-section. The variational iteration method is applied to the problem of determination of critical buckling loads for Euler columns with variable cross-sections in the paper of Coskun and Atay [10]. Coskun [11] used the homotopy perturbation method for determination of critical buckling loads for Euler columns of variable flexural stiffness with a continuous elastic restraint. Singh and Li [12] studied the stability of axially FG tapered beams through modeling non-uniform beams as an assemblage of several uniform segments. As a result, the system of transcendental equations is solved for determining critical buckling load. The

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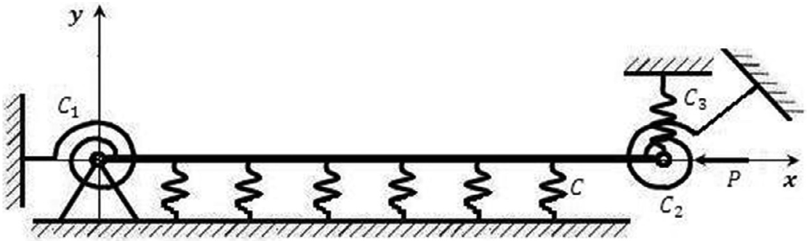

Fig. 1. The beam on elastic foundation with general elastically end constraints.

proposed method is known as TEP (transcendental eigenvalue problem) method. The results of the TEP method are compared with the results of finite element (FEM) and finite difference (FDM) methods [12].

Huang and Li [13,14] studied buckling instability of non-uniform and axially FG columns with or without elastic foundation by transforming the governing differential equation to Fredholm integral equation. For similar problems Huang and Luo [15] applied simple power series expansion of the mode shapes and transformed the governing differential equations with variable coefficients to system algebraic equations. Elishakoff in [16] determined the grading function $D(x)$ corresponding to the buckling mode in the form of the polynomial function for a column made of axially FG material.

Structural analysis (free vibration and stability) of axially functionally graded tapered Euler–Bernoulli beams is studied using finite element method by Shahba et al. in [17]. In [18] Shahba and Rajasekaran introduced the differential transform element method (DTEM) and differential quadrature element method lowest-order (DQEL) to perform structural analysis of tapered Euler–Bernoulli beam made of axially FG materials. The same problems are solved by Rajasekaran in [19] using dynamic stiffness based approach and differential transformation. Tomasiello [20] considered a numerical Picard-like method to analyze the stability of columns; that method combines successive approximations with differential quadrature method (DQM).

Recently, Yilmaz et al. [21] applied localized differential quadrature method (LDQM) to the buckling analyses of non-uniform axially FG columns with elastic restraints for different boundary conditions. The paper [21] gives high-accuracy numerical results for critical buckling loads for various cases of axially FG columns with elastic restraints.

During last decade, the Haar wavelet based discretization methods (HWDM) have been developed for numerical analysis of the various solid mechanics problems [22]. The buckling of elastic beams with variable cross section is studied using HWDM in [22,23].

The current study proposes the numerical method for analysis of buckling of axially FG beams on elastic foundation under concentrated compressive force. The boundary conditions are considered in general form covering different classical supporting conditions. The current approach can be regarded as method of initial parameters in differential form. The proposed approach allows satisfying automatically the boundary conditions on the left end of the beam (initial parameters). The solution of the posed boundary value problem is reduced to the iterative solution of two initial value problems and homogeneous linear algebraic system of order two. The initial value problems are solved by the fourth-order Runge–Kutta method in each iteration. Refined solutions are obtained by applying Richardson extrapolation method [24,25].

2. Formulation of the problem

Let us consider a functionally graded (FG) initially rectilinear beam on a Winkler elastic foundation under compressive force P. Using the Euler–Bernoulli law of bending states, the equation of the buckling of this problem is [1–4]:

$$(D(x)y'')'' + Py'' + Cy = 0, \quad x \in [0, L], \tag{1}$$

where x, y and L are axial coordinate, the transverse deflection and length of the beam, respectively, C is known as the modulus or the stiffness of the foundation. $D(x) = E(x)I(x)$ is flexural rigidity which depends upon the axially graded Young's modulus of elasticity $E(x)$, and/or spatially varying geometry with the moment of cross-section inertia $I(x)$. The general boundary conditions are given by:

$$\begin{aligned} &D(0)y''(0) = C_1 y'(0), \quad y(0) = 0, \\ &D(L)y''(L) = -C_2 y'(L), \qquad (D(L)y''(L))' + Py'(L) = C_3 y(L). \end{aligned} \tag{2}$$

Here C_1 is rotational spring constant on left end of the beam ($x = 0$), and C_2, C_3 are rotational and translational spring constants on the right end of the beam ($x = L$), as shown in Fig. 1. The different classical boundary conditions can be obtained for the values of spring constants $C_i = 0$ or $C_i = \infty$, i = 1, 2, 3. For example, we have $C_1 = \infty$, $C_2 = 0$, $C_3 = \infty$ for clamped-pinned boundary conditions.

Instead of spring constants C_i let us introduce new parameters of stiffness $c_i = C_i / (C_i + 1)$, $i = 1, 2, 3$. Thus, when parameter C_i takes values from 0 *to* ∞, the parameter c_i varies from 0 to 1. Using an inverse transformation $C_i = c_i / (1 - c_i)$, $i = 1, 2, 3$ the boundary conditions (2) can be rewritten as:

$$\begin{aligned} &y(0) = 0, \quad c_1 y'(0) - (1 - c_1)(D(0)y''(0)) = 0, \\ &c_2 y'(L) + (1 - c_2)(D(L)y''(L)) = 0, \quad c_3 y(L) + (c_3 - 1)((D(L)y''(L))' + Py'(L)) = 0. \end{aligned} \tag{3}$$