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Stability of nonlinear Caputo fractional differential equations☆

Kewei Liu [a,b,*], Wei Jiang [b]

[a] School of Mathematics, Hefei University of Technology, Hefei, China
[b] School of Mathematical Sciences, Anhui University, Hefei, China

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ABSTRACT

In this paper, by using the fractional differential inequality, we obtain a Caputo type fractional comparison result. Further, We investigate the stability and instability of nonlinear Caputo fractional differential equations by using the Caputo type fractional comparison principle.

1. Introduction

In the last decades, fractional calculus and fractional order differential equations have attracted a great attention. Fractional order systems have found many applications in various domains such as heat transfer, viscoelasticity, electrical circuit, electro-chemistry, dynamics, economics and control. For details and examples, see [1–4] and the references therein.

Stability analysis is always one of the most important issues for differential equations, although this problem has been investigated over many years. Recently, stability of fractional differential systems has attracted increasing interest. The earliest study on stability of fractional differential equations started in [5]. Since then, many researchers have done studies on the stability. And some basic results on the stability and other related problems are obtained [6–15]. For more details about the stability results and the methods available to analyze the stability of fractional differential equations, the reader may refer to the recent survey papers [16,17] and the references therein.

For the nonlinear fractional differential systems, the stability analysis is much more difficult and a few results are available in [9–13,15]. Motivated by Li et al. [10,11], and Lakshmikantham and Vatsala [18,19], in this paper, we consider the stability of nonlinear fractional differential equations with the Caputo derivative. By using fractional differential inequalities, we develop a fractional comparison result for Caputo fractional differential equation. Finally, by using the fractional comparison principle, we investigate the stability and instability of nonlinear Caputo fractional differential equations.

This paper is organized as follows: In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we discuss a result of fractional differential inequalities and develop a Caputo type fractional comparison principle. In Section 4, we consider the stability and instability of nonlinear Caputo fractional differential equations. At last, some examples are given to illustrate our results.

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* Corresponding author at: School of Mathematics, Hefei University of Technology, Hefei, China. Tel.: +86 0551 62902673.
E-mail addresses: liu_kwei@163.com, kweiliu@163.com (K. Liu), jiangwei@ahu.edu.cn (W. Jiang).

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2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts needed here. Throughout this paper, let $C([t_0, T], \mathbb{R}^n)$ be Banach space of all continuous functions from$[t_0, T]$ into $\mathbb{R}^n$ with the norm $\|x\|_C = sup\{\|x(t)\| : t \in [t_0, T]\}$ for $x \in C([t_0, T], \mathbb{R}^n)$.

Let us recall the following known definitions. For more details about fractional order detivative/integral, the reader may refer to [1–4].

Definition 2.1. The fractional order integral of a function $f : [t_0, \infty) \to \mathbb{R}$ of order $\gamma \in \mathbb{R}^+$ is defined by

$$I_{t_0}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} f(s) ds$$

provided the integral exists, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. For a function f given on the interval $[t_0, \infty)$, the γ order Riemann–Liouville fractional derivative of f is defined by

$$D_{t_0}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-\gamma-1} f(s) ds, n-1 < \gamma < n, n \in \mathbb{N}$$

provided the integral exists.

Definition 2.3. For a function f given on the interval $[t_0, \infty)$, the γ order Caputo fractional derivative of f is defined by

$${}^{c}D_{t_0}^{\gamma} f(t) = D_{t_0}^{\gamma} \left[f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)(t-t_0)^i}{i!} \right], n-1 < \gamma < n, n \in \mathbb{N}$$

provided the integral exists.

Definition 2.4. The Mittag-Leffler function in two parameters is defined as $E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+\beta)}, z \in \mathbb{C}$, where $\alpha > 0$, $\beta > 0$, $\mathbb{C}$ denotes the complex plane.

Remark 2.5. $E_{\alpha,1}(\lambda z^{\alpha}) = E_{\alpha}(\lambda z^{\alpha}) = \sum_{i=0}^{\infty} \frac{\lambda^i z^{i\alpha}}{\Gamma(i\alpha+1)}, \lambda, z \in \mathbb{C}$.

Definition 2.6. If a continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing, and $\varphi(0) = 0$, we call φ a K-class function, denoted by $\varphi \in K$. Here $\mathbb{R}^+ = [0, \infty)$.

Definition 2.7. Let $x = 0$ be the zero solution of ${}^{c}D_{t_0}^{\alpha} x(t) = f(t, x)$ with $\alpha \in (0, 1)$ and $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

(1) The zero solution $x = 0$ is said to be stable if for $\forall \varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that $\|x(t_0)\| < \delta(\varepsilon)$, implies $\|x(t)\| \le \varepsilon$ for $t \ge t_0$. The zero solution $x = 0$ is said to be unstable, if $\exists \varepsilon_0 > 0$, $\forall \delta > 0$, $\exists x(t_0)$, $\|x(t_0)\| < \delta$, but $\exists t_1 \ge t_0$ such that $\|x(t_1)\| \ge \varepsilon$.

(2) The zero solution $x = 0$ is said to be asymptotically stable if it is stable and $\lim_{t\to+\infty} x(t) = 0$.

Definition 2.8. Let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Further, suppose that $x = 0$ is the zero solution of ${}^{c}D_{t_0}^{\alpha} x(t) = f(t, x)$.

(1) The zero solution $x = 0$ is said to be Mittag-Leffler stable if

$$\|x(t)\| \le [m(x(t_0)) E_{\alpha}(-\lambda(t-t_0)^{\alpha})]^q.$$

(2) The zero solution $x = 0$ is said to be generalized Mittag-Leffler stable if $\|x(t)\| \le [m(x(t_0))(t-t_0)^{-\gamma} E_{\alpha,1-\gamma}(-\lambda(t-t_0)^{\alpha})]^q$.

Here $\alpha \in (0, 1)$, $\lambda > 0$, $q > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{D}$ with Lipschitz constant m_0.

3. Caputo type fractional comparison principle

Consider the stability of the following Caputo fractional differential equation

$${}^{c}D_{t_0}^{\alpha} x(t) = f(t, x), \tag{1}$$

with the initial condition $x(t_0) = x_0$, where $0 < \alpha < 1$, $f \in C(\mathbb{R} \times \mathbb{D}, \mathbb{R}^n)$, $f(t, 0) \equiv 0$, $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. In this section, we will present the general Caputo type fractional comparison principle, which is fundamental in the investigation of the stability of fractional differential systems. Here, we always assume there exists a unique continuously differentiable solution $x(t)$ to (1) with the initial condition x_0.

For comparison, consider a scale differential equation

$${}^{c}D_{t_0}^{\alpha} u(t) = g(t, u), \tag{2}$$