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Ergodicity of large scale stochastic geophysical flows with degenerate Gaussian noise☆

Lian Yang[a], Xueke Pu[b,*]

[a] School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China
[b] College of Mathematics and Statistics, Chongqing University, Chongqing 401331, PR China

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ABSTRACT

This paper considers ergodicity of the stochastic quasigeostrophic equations driven by degenerate Gaussian noise. Uniqueness of invariant measures is shown by proving the asymptotically strong Feller property of the probability transition semigroups.

1. Introduction

The simplified models of geophysical flows, derived at asymptotically high rotation or small Rossby number, capture the features of large scale phenomena and filter out undesired high-frequency oscillations in geophysical flows. An important and extensively studied deterministic model is given by the following quasi-geostrophic equation

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi,$$

where $\psi(t, x, y)$ is the stream function, $\beta \geq 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant and $r > 0$ is the Ekman dissipation constant. Moreover, $J(f, g) = \nabla^{\perp} f \cdot \nabla g = f_x g_y - f_y g_x$ denotes the Jacobian operator.

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* Corresponding author.
E-mail address: xuekepu@cqu.edu.cn (X. Pu).

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In the past decades, the randomly forced quasigeostrophic flow model has been widely used to study various phenomena in geophysical flows under uncertain and wind forcing. See [1,2,3] and the references therein, to list only a few. This leads us to consider the following stochastically forced quasigeostrophic flow model with random forcing

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + \text{wind forcing}, \tag{1.1}$$

where $(x, y) \in D$, and D is a bounded smooth domain in $\mathbf{R}^2$. Introducing the relative vorticity $u = \Delta\psi$, the quasigeostrophic equation can be written as

$$u_t + J(\psi, u) + \beta\psi_x = \nu\Delta u - ru + \text{wind forcing}. \tag{1.2}$$

In the following, we also write $\psi = \psi(u) = -(-\Delta)^{-1}u$ to emphasize the dependence of ψ on u. The boundary conditions are given as in Pedlosky [4] $\psi = u = 0$ on ∂D. The Poincare inequality then holds with constant assumed to be 1, for simplicity. The initial condition $u_0 \in L^2(D)$ is also imposed.

The random wind forcing is modeled by the white in time Gaussian noise $Q\dot{W}$, where $W(t, x, y)$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the covariance operator $Q : L^2 \to L^2$ is a nonnegative and symmetric linear continuous operator to be specified below. The stochastic quasigeostrophic equation can then be written as

$$du = (\nu\Delta u - ru - J(\psi, u) - \beta\psi_x)dt + QdW. \tag{1.3}$$

Ergodicity is an important issue in long time behaviors of stochastic partial differential equations, which implies that the time average tends to the statistical ensemble average, as time goes to infinity. Therefore, it is interesting to study ergodicity of the above quasigeostrophic flow (1.2). Brannan et al. studied the existence and uniqueness of solutions in $C([0, T]; L^2(D))$, under very mild solutions. Under the injective condition of Q, i.e., $Q : L^2(D) \to L^2(D)$ is one-to-one, together with other conditions, Duan and Goldys showed the ergodicity of (1.3), using Doob's theorem, by combining the strong Feller and irreducibility properties [3]. In other words, when the noise is non-degenerate, they showed ergodicity of the stochastic quasigeostrophic equation (1.3). But their results cannot cover the more interesting case when the noise is degenerate, that is when the noise drives the system only on a finite Fourier modes.

The degenerate noise case is more interesting and attracted many researchers in SPDEs in the past decades. In particular, Hairer and Mattingly [5] studied the ergodicity of the 2D stochastic Navier–Stokes equation driven by degenerate noise, by proposing the important idea of asymptotically strong Feller property. For ergodicity of the stochastic fractional surface quasigeostrophic equation with degenerate noise, see also [2].

In this paper, we will show ergodicity for the quasigeostrophic equation (1.3) under degenerate noise. Specifically, we let $\{e_1, e_2, \ldots\}$ be the normalized eigenfunctions corresponding to the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ of $-\Delta$ from $L^2(D)$ to $L^2(D)$ with domain $D(-\Delta) = H^2(D) \cap H_0^1(D)$. We set $QW(t) = \sum_{k=1}^{N} \alpha_k\beta_k(t)e_k$ for some integer $N > 0$, for $t \geq 0$, where $\{\beta_k(t)\}$ is a class of independent real-valued Brownian motions. Without loss of generality, we set all the α_k's to be 1. The noise is degenerate in the sense that it drives the system only in the first N Fourier modes. The existence and uniqueness of solutions in $C([0, T]; L^2(D))$ and existence of invariant measures are given respectively in [1] and [3].

The main result in this paper is stated in the following.

Theorem 1.1. *There exists $N > 0$ such that there exists a unique invariant probability measure for the stochastic quasigeostrophic equation (1.3) in $L^2(D)$.*

The proof is based on the idea of asymptotically strong Feller property of Hairer and Mattingly [5]. In the next section, we will show this property in Proposition 2.1, which combined with a support property in Proposition 2.2 yields the result. In particular, we will need the following basic estimates in [6].