Applied Mathematics Letters 61 (2016) 163–166

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Extinction for the super diffusion equation with a nonlocal absorption and a gradient source ☆

Haixing Zhang, Yujuan Chen *

School of Sciences, Nantong University, Nantong 226019, PR China

ARTICLE INFO

Article history:
Received 20 April 2016
Received in revised form 1 June 2016
Accepted 1 June 2016
Available online 9 June 2016

Keywords:
Super diffusion
Gradient source
Absorption term
Extinction in finite time

ABSTRACT

In this article, the authors establish the conditions for the extinction of solutions in finite time of the super diffusion equation $u_t = \Delta u^m + \lambda|\nabla u|^q - a\int_\Omega u^p dx, m < -1$, in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N > 2$. It is shown that if $\frac{2}{3-m} \le q < \frac{m}{m-1}, p < 1 + m$, for sufficiently small initial data u_0, the weak solution $u(x, t)$ vanishes in finite time.

1. Introduction

In this article, we investigate the extinction property of the solutions for the following super diffusion equation with an absorption term and gradient source

$$\begin{cases} u_t = \Delta u^m + \lambda|\nabla u|^q - a\displaystyle\int_\Omega u^p dx, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $m < -1, q, a, \lambda > 0, p < 0, \Omega$ is a bounded domain in $\mathbb{R}^N (N > 2)$ with smooth boundary $\partial\Omega, u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonnegative function.

As we know, when $0 < m < 1$ the type of diffusion in (1.1) is called fast diffusion, when $m > 1$, the type is called slow diffusion and when $m < 0$, we often name it super diffusion and it arises from a number of concrete problems, including diffusion in plasma [1,2] and heat conduction in solid hydrogen [3] etc.

The subject of extinction or nonextinction has received much attention starting with the pioneering work of Kalashnikov [4] who considered the Cauchy problem of equation $u_t = \Delta u - u^p$. Later, many authors

☆ This work was supported by PRC Grants NSFC 11371370.

* Corresponding author.
E-mail address: cyj_nt@163.com (Y. Chen).

http://dx.doi.org/10.1016/j.aml.2016.06.003
0893-9659/

became interested in it (see [5–9] and references therein). As for the case of fast diffusion ($0 < m < 1$), Li et al. [5] gave some necessary and sufficient conditions of extinction for the solutions to the parabolic initial and boundary value problem of equation $u_t = \Delta u^m + \lambda u^p$ in a bounded domain Ω, and proved that if $p > m$, the solutions to the problem with small initial data vanish in finite time and if $p < m$, the maximal solution is positive in Ω for all $t > 0$. For the critical case $p = m$, the first eigenvalue of $-\Delta$ in Ω plays an important role. Recently, Mu et al. [6] studied the equation $u_t = \Delta u^m + \lambda|\nabla u|^p$, and proved that (i) if $m < p \leq \frac{2}{3-m}$, the weak solution with small initial data vanishes in finite time, (ii) if $p < m$, the weak solution $u(x,t)$ cannot vanish in finite time for any nonnegative initial date u_0 with λ being sufficiently large, and (iii) if $p = m \leq \frac{2}{3-m}$, the solution vanishes in finite time for λ sufficiently small.

As for the nonlocal sources, Han et al. [7] considered the equation $u_t = \Delta u^m + a\int_\Omega u^q(y,t)dy$, Xu et al. [8] investigated $u_t = d\Delta u^m + \lambda\int_\Omega u^q(x,t)dx - \beta u^k$. The authors obtained some sufficient conditions for the extinction of nonnegative weak solutions and also obtained the decay estimates.

Motivated by those works above, we think that there will be some new conditions to cause the extinction to happen if an absorption term and a nonlinear gradient source exist together. Moreover, it is observed that when extinction is concerned, the majority of the work is concerned with the case of fast diffusion $0 < m < 1$. Here we study a kind of a super diffusion case. For our convenience, we first define some sets as follows:

$$E = \{u : u^m, u^p, u \in L^2(\Omega_T); \nabla u \in L^{2q}(\Omega_T)\},$$
$$E_0 = \{\xi \in L^2(\Omega_T) : \xi_t, \Delta\xi \in L^2(\Omega_T); \xi|_{\partial\Omega_T} = 0\}.$$

It is well known that the problem (1.1) has no classical solution in general. We need to consider its weak solutions which is defined as follows.

Definition 1.1. For any $T \geq 0$, a function $u(x,t) \in E$ is called a weak solution to the (1.1), if the following equalities hold for $0 < t_1 < t_2 < T, 0 \leq \xi \in E_0$

$$\begin{aligned}
&\int_\Omega u(x,t_2)\xi(x,t_2)dx - \int_\Omega u(x,t_1)\xi(x,t_1)dx \\
&= \int_{t_1}^{t_2}\int_\Omega \{u\xi_t + u^m\Delta\xi\}dxdt + \lambda\int_{t_1}^{t_2}\int_\Omega |\nabla u|^q\xi dxdt - a\int_{t_1}^{t_2}\int_\Omega\left(\int_\Omega u^p dx\right)\xi dxdt, \\
&u(x,0) = u_0(x) \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

The main result is presented as follows.

Theorem 1.1. *Assume that $m < -1$, $\frac{2}{3-m} \leq q < \frac{m}{m-1}$ and $p < 1+m$. Then, for sufficiently small initial data u_0, the weak solution $u(x,t)$ of (1.1) satisfies*

$$\begin{cases} \|u\|_{1+m} \leq \|u_0\|_{1+m}\left[1 - \dfrac{(1-m)C_2 t}{\|u_0\|_{1+m}^{1-m}}\right]_+^{\frac{1}{1-m}}, & t \in (0, T_1), \\ u(x,t) \text{ does not exist}, & t \in [T_1, +\infty); \end{cases} \tag{1.2}$$

where C_2 and T_1 are defined by (2.11) and (2.12), respectively, $w_+ = max\{w, 0\}$. In this sense, we call u extinct in finite time T_1.

2. Proof of theorem

Proof of Theorem 1.1. We apply the energy estimate method to prove it. First of all, multiplying the first equation of the problem (1.1) by u^{s-1} and integrating over Ω, we can obtain the following