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A simple kinetic equation of swarm formation: Blow-up and global existence

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ABSTRACT

In the present paper we identify both blow-up and global existence behaviors for a simple but very rich kinetic equation describing of a swarm formation.

1. Introduction

In paper [1] a model of swarming behavior of an individual population was proposed and studied. The main aim was the macroscopic (*hydrodynamic*) limit. The mathematical structure that was proposed seems very reach and interesting from mathematical point of view. Let $f = f(t, x, v)$ be a probability density (p.d.) of individuals at time $t \geq 0$ and position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{V}$; $\mathbb{V} \subset \mathbb{R}^d$, the set of velocities of the individuals, is a bounded domain. The evolution of populations at the mesoscopic scale is defined by the

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nonlinear integro-differential Boltzmann-like equation, see [1],

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \frac{1}{\varepsilon} Q[f](t,x,v)$$
$$= \frac{1}{\varepsilon} \int_{\mathbb{V}} \Big(T[f(t,x,.)](w,v) f(t,x,w) - T[f(t,x,.)](v,w) f(t,x,v) \Big) \mathrm{d}w \tag{1}$$

with the initial data $f(0,x,v) = f_0(x,v)$. The parameter ε corresponds to the Knudsen number and the macroscopic limit is defined by $\varepsilon \to 0$. The nonlinear operator Q describes interactions between individuals. The turning rate $T[f](v,w)$ measures the probability for an individual with velocity v to change velocity into w. Macroscopic limit for a simpler (two-velocities) kinetic equation was studied in [2] (see also [3]). In the context of modeling of preferential choice one should mention the paper [4] where a collision model was proposed describing ant trail formation.

In Ref. [1] the following general nonlinear case

$$T[f(t,x,.)](v,w) = \sigma \beta(v,w) f^{\gamma_{\rho,x}}(t,x,w), \tag{2}$$

was considered, where the interaction rate β, the attractiveness coefficient γ, and σ characterize the interaction between the individual agents. Parisot and Lachowicz [1] proposed results of global existence in the space homogeneous case for any set of collision parameters σ and γ except the so-called positive gregarious interaction, i.e. $\sigma = 1$ and $\gamma > 1$. The aim of the present paper is the analysis of simpler (but still rich) equation in this case. More general equation and some details of the present approach will be given in Ref. [5].

2. Mathematical analysis of the space homogeneous case

We focus on the space homogeneous case, i.e. all functions and parameters are assumed to be independent of x. Moreover we assume that σ, γ, β are constants: $\sigma = \beta = 1$, and $\gamma > 1$.

Throughout the paper the L^p–norm in the velocity space $\mathbb{V}$ is denoted by $\|\phi\|_p = \left(\int_{\mathbb{V}} \phi^p \mathrm{d}v\right)^{\frac{1}{p}}$. It is easy to see that any solution preserves the nonnegativity of the initial datum and the L^1–norm of the nonnegative initial datum. Therefore Eq. (1) can be simplified to the following equation

$$\partial_t f = f^\gamma - \|f\|_\gamma^\gamma f, \quad \text{with} \quad f(0,v) = f_0(v), \quad t \geq 0, \ v \in \mathbb{V}. \tag{3}$$

Let $z(t) = \int_0^t \|f(s,\,.)\|_\gamma^\gamma \mathrm{d}s$. It fulfills

$$\mathrm{d}_t z(t) = e^{-\gamma z(t)} \int_{\mathbb{V}} \left(f_0^{1-\gamma} - (\gamma-1) \int_0^t e^{-(\gamma-1) z(s)} \mathrm{d}s \right)^{-\frac{\gamma}{\gamma-1}} \mathrm{d}v. \tag{4}$$

Eq. (4) determines global existence or blow-up for Eq. (3). Let $u(t) = \int_0^t e^{-(\gamma-1)z(s)} \mathrm{d}s$. The function $u = u(t)$ is increasing and (as we will see) concave. A blow-up occurs for $T > 0$ such that

$$(\gamma-1) \|f_0\|_\infty^{\gamma-1} u(T) = 1. \tag{5}$$

The ODE for u reads $\mathrm{d}_t u(t) = e^{-(\gamma-1)z(t)}$ and we have

$$\mathrm{d}_t^2 u = -(\gamma-1) e^{-(\gamma-1)z} \mathrm{d}_t z$$
$$= -(\gamma-1) \mathrm{d}_t u \, (\mathrm{d}_t u)^{\frac{\gamma}{\gamma-1}} \int_{\mathbb{V}} \left(f_0^{1-\gamma} - (\gamma-1) u \right)^{-\frac{\gamma}{\gamma-1}} \mathrm{d}v.$$

By integration we obtain

$$\mathrm{d}_t u = \left(\int_{\mathbb{V}} f_0(v) \Big(1 - (\gamma-1) f_0^{\gamma-1} u \Big)^{\frac{1}{1-\gamma}} \mathrm{d}v \right)^{1-\gamma}. \tag{6}$$