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Energy dissipation in the Smagorinsky model of turbulence

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ABSTRACT

The Smagorinsky model often severely over-dissipates flows and, consistently, previous estimates of its energy dissipation rate blow up as $Re \to \infty$. This report estimates time averaged model dissipation, $\langle \varepsilon_S \rangle$, under periodic boundary conditions as

$$\langle \varepsilon_S \rangle \leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L} + \frac{32}{27}C_S^2\left(\frac{\delta}{L}\right)^2\frac{U^3}{L},$$

where U, L are global velocity and length scales and $C_S \simeq 0.1, \delta < 1$ are model parameters. Thus, in the absence of boundary layers, the Smagorinsky model does not over dissipate.

1. Introduction

The Smagorinsky model, used for turbulent flow simulations, [1,2], is given by

$$u_t + u \cdot \nabla u - \nu \triangle u - \nabla \cdot \left((C_S \delta)^2 |\nabla u| \nabla u \right) + \nabla p = f(x) \ \&\ \nabla \cdot u = 0. \tag{1}$$

It is related to the Ladyzhenskaya model [3–5] and the von Neumann–Richtmyer method for shocks. In (1), ν is the kinematic viscosity, $\delta \ll 1$ is a model length scale, the Reynolds number is $Re = LU/\nu$ where U, L are given in (3), and $C_S \simeq 0.1$. Experience with the model, e.g., [2], indicates it over dissipates, often severely, consistent with estimates of model energy dissipation rates for *shear flows* in [6]. Model refinements aim at reducing model dissipation occur as early as 1975, [7], and currently, [2] Section 4.3 p. 117 onwards, include approaches such as dynamic parameter selection, structural sensors, the accentuation technique and damping functions. Perhaps surprisingly, Theorem 1 below establishes that *the Smagorinsky model does not over dissipate in the absence of boundary layers.*

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Let $\Omega = (0, L_\Omega)^3$. For $\phi = u,\ u_0,\ f,\ p$ impose periodicity

$$\phi(x + L_\Omega e_j, t) = \phi(x,t) \quad j = 1,2,3 \quad \text{and} \quad \int_\Omega \phi dx = 0. \tag{2}$$

The data $u_0(x), f(x)$ are smooth, periodic, have zero mean and satisfy $\nabla \cdot u_0 = 0$ and $\nabla \cdot f = 0$. The model energy dissipation rate from (5) below is

$$\varepsilon_S(u) := |\Omega|^{-1} \int_\Omega \nu |\nabla u(x,t)|^2 + (C_S\delta)^2 |\nabla u(x,t)|^3 dx$$

and the long time average of a function $\phi(t)$ is defined by

$$\langle \phi \rangle := \limsup_{T\to\infty} \frac{1}{T} \int_0^T \phi(t) dt.$$

The estimate below, $\langle \varepsilon_S \rangle \simeq U^3/L$, is consistent as $Re \to \infty, \delta \to 0$ with both phenomenology, [8], and the rate proven for the Navier–Stokes equations in [9–11].

Theorem 1. $\langle \varepsilon_S \rangle$ *satisfies: for any* $0 < \alpha < \frac{2}{3}$,

$$\langle \varepsilon_S(u) \rangle \le \frac{1}{1-\alpha}\frac{U^3}{L} + \frac{1}{4\alpha(1-\alpha)}\mathrm{Re}^{-1}\frac{U^3}{L} + \frac{4}{27(1-\alpha)\alpha^2} C_S^2 \left(\frac{\delta}{L}\right)^2 \frac{U^3}{L}.$$

1.1. Related work

The energy dissipation rate is a fundamental statistic of turbulence, e.g., [8,12]. In 1992 Constantin and Doering [9] established a direct link between phenomenology and NSE predicted energy dissipation. This work builds on [13,14] (and others) and has developed in many important directions subsequently.

2. Proof of Theorem 1

Let $\|\cdot\|, (\cdot,\cdot), \|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm, inner product and $L^p(\Omega)$ norm. The force, large scale velocity and length scales, F, U, L, are

$$F = \left(\frac{1}{|\Omega|}\|f\|^2\right)^{\frac{1}{2}}, \qquad U = \left\langle \frac{1}{|\Omega|}\|u\|^2 \right\rangle^{\frac{1}{2}} \text{ and} \tag{3}$$
$$L = \min\left\{ |\Omega|^{\frac{1}{3}}, \frac{F}{\|\nabla f\|_\infty}, \frac{F}{(\frac{1}{|\Omega|}\|\nabla f\|^2)^{\frac{1}{2}}}, \frac{F}{(\frac{1}{|\Omega|}\|\nabla f\|_3^3)^{\frac{1}{3}}} \right\}.$$

It is easy to check that L has units of length and satisfies

$$\|\nabla f\|_\infty \le \frac{F}{L}, \qquad \frac{1}{|\Omega|}\|\nabla f\|^2 \le \frac{F^2}{L^2} \quad \text{and} \quad \frac{1}{|\Omega|}\|\nabla f\|_3^3 \le \frac{F^3}{L^3}. \tag{4}$$

Solutions to (1) (2) are known, e.g., [4,3,5], to be unique strong solutions and satisfy the energy equality

$$\frac{1}{2|\Omega|}\|u(T)\|^2 + \int_0^T \varepsilon_S(u) dt = \frac{1}{2|\Omega|}\|u_0\|^2 + \int_0^T \frac{1}{|\Omega|}(f, u(t))\, dt. \tag{5}$$

Here $\varepsilon_S(u) = \varepsilon_0(u) + \varepsilon_\delta(u), \varepsilon_0 = |\Omega|^{-1}\nu\|\nabla u(t)\|^2$ and $\varepsilon_\delta = |\Omega|^{-1}(C_S\delta)^2\|\nabla u(t)\|_3^3$. From (5) and standard arguments it follows that

$$\sup_{t\in(0,\infty)} \|u(t)\|^2 \le C(data) < \infty \quad \text{and} \quad \frac{1}{T}\int_0^T \varepsilon_S(u) dt \le C(data) < \infty. \tag{6}$$