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Superlinear discrete problems

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ABSTRACT

The aim of this paper is to present an existence result of two positive solutions for a nonlinear difference problem by variational methods. The conclusion is achieved by assuming, together with the super-linearity at infinity, a suitable algebraic condition on the nonlinear term, which is more general than the sub-linearity at zero.

1. Introduction

Consider the following Dirichlet discrete boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f_k(u(k)), \quad k \in [1,N], \\ u(0) = u(N+1) = 0, \end{cases} \qquad (D_\lambda^f)$$

where N is a positive integer, $[1,N]$ denotes the discrete interval $\{1,\dots,N\}$, and, for every $k \in [1,N]$, $\Delta u(k-1) := u(k) - u(k-1)$ is the forward difference operator, $\Delta^2 u(k-1) := u(k+1) - 2u(k) + u(k-1)$ is the second order difference operator, $f_k : \mathbb{R} \to \mathbb{R}$ are the components of a continuous function $\underline{f} : \mathbb{R} \to \mathbb{R}^N$ and λ is a positive real parameter. For general references on difference equations and their applications, we refer the reader to monographs [1,2], while for recent results we cite [3–14] and references therein.

The aim of this note is to establish the existence of two positive solutions for (D_λ^f) under suitable assumptions on the nonlinearity $\underline{f}$. Here, as an example, we present the following special case of our main result.

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Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty; \tag{1.1}$$

and

$$\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

Then, for each $\lambda \in]0, \frac{2}{N(N+1)} \sup_{c>0} \frac{c^2}{\max_{s\in[0,c]} \int_0^s f(t)dt}[$, *the problem*

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{D_λ^f}$$

admits at least two positive solutions.

We recall that the study of the existence of two non trivial solutions for nonlinear problems depending on a parameter is a fundamental argument also for differential equations. In particular, Amann has proved in [15] the existence of two positive solutions for a two-point boundary value problem by assuming, besides to (1.2), that $f(0) > 0$ which is less general than (1.1), while Crandall and Rabinowitz have established in [16] the same conclusion for elliptic Dirichlet problems assuming, together with $f(0) > 0$, the celebrated condition of Ambrosetti–Rabinowitz which is, in turn, less general than (1.2). Finally, we also note that Ambrosetti, Brezis and Cerami, in [17], have proved such a conclusion, again for elliptic Dirichlet problems, when the nonlinear term is given by combined effects of concave and convex nonlinearities, that is, $f(t) = |t|^q + \mu|t|^s$, $0 < s < 1 < q$, $\mu > 0$, in order to study a case when $f(0) = 0$. For more details on such differential problems we refer to [18]. Here, we wish to point out that for the difference equations we can obtain the same conclusion under more general assumptions on the nonlinearity than those required for differential problems, as Theorem 1.1 highlights. Indeed, this can be applied also to nonlinearities for which $f(0) = 0$ (see Example 2.1) and the Ambrosetti–Rabinowitz condition does not hold (see Example 2.2). Also, with regard to the results in the literature on difference equations, our hypotheses are the most general. In particular, Theorem 1.1 improves [19, Theorem 2.1] where $f(0) > 0$ is required. Moreover, the Ambrosetti–Rabinowitz condition, usually invoked also for nonlinear discrete problems (see, for instance, [20–23]), as already said, here is not assumed.

The main result of the paper is Theorem 2.1. The algebraic assumptions from Theorem 2.1 are more general than the sublinearity at zero and the superlinearity at infinity in Theorem 1.1 and also include, in particular, the linearity at zero and at infinity, as easy Example 2.3 shows. Our approach is based on a two non-zero critical points theorem established in [18], which is a suitable combination of the classical Ambrosetti–Rabinowitz theorem (see [24]) and a local minimum theorem established in [25] (see also [26]). We recall it here for the reader's convenience.

Theorem 1.2. *Let X be a real Banach space and let $\varphi, \psi : X \to \mathbb{R}$ be two functionals of class C^1 such that* $\inf_X \varphi = \varphi(0) = \psi(0) = 0$. *Assume that there exist $r \in \mathbb{R}$ and $w \in X$, with $0 < \varphi(w) < r$, such that*

$$\frac{\sup_{x\in\varphi^{-1}(]-\infty,r])} \psi(x)}{r} < \frac{\psi(w)}{\varphi(w)}$$

and, for each $\lambda \in \Lambda :=]\frac{\varphi(w)}{\psi(w)}, \frac{r}{\sup_{x\in\varphi^{-1}(]-\infty,r])} \psi(x)}[$, *the functional $\mathcal{I}_\lambda = \varphi - \lambda\psi$ satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda$, the functional $\mathcal{I}_\lambda$ admits at least two non-zero critical points $x_{\lambda,1}, x_{\lambda,2}$ such that $\mathcal{I}(x_{\lambda,1}) < 0 < \mathcal{I}(x_{\lambda,2})$.